

Exponential number of equilibria and depinning threshold for a directed polymer in a random potential

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Using the Kac-Rice approach, we show that the mean number $\langle \mathcal{N}_{\text{tot}} \rangle$ of all possible equilibria of an elastic line (directed polymer), confined in an harmonic well and submitted to a quenched random Gaussian potential, grows exponentially $\langle \mathcal{N}_{\text{tot}} \rangle \sim \exp(rL)$ with its length L . The growth rate r is found to be directly related to the fluctuations of the Lyapunov exponent of an associated Anderson localization problem of a 1d Schrödinger equation in a random potential. For strong confinement, the rate r is small and given by a non-perturbative (instanton) contribution to the Lyapunov exponent. For weak confinement, the rate r is found proportional to the inverse Larkin length of the pinning theory. As an application, identifying the depinning with a landscape "topology trivialization" phenomenon, we obtain an upper bound for the depinning threshold f_c , in presence of an applied force.

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Various aspects of the behaviour of a directed polymer, i.e. an elastic line, in a quenched random potential keep attracting permanent research efforts of both physicists and mathematicians for more than three decades. Among other applications, it was at the center of attention as a model for vortex lines in superconductors, leading to important developments in the physics of pinning (see [1, 2] for reviews). Its connection to the Kardar-Parisi-Zhang growth (see [3] for review of earlier works) led to a recent outburst of interest, and it was shown that the probability density of the free energy for a long polymer converges to the famous Tracy-Widom distribution [4–7], extending the result for the ground state energy [8].

In this Letter we address a somewhat different aspect and consider the problem of counting the total number of *equilibria* for a directed polymer (DP), in general harmonically confined, immersed in a random potential. Those are defined as the stationary points (minima, maxima, or saddles) of an energy functional (see below). From a broader perspective, describing the statistical structure of the stationary points of random landscapes and fields of various types is a rich problem of intrinsic current interest in various areas of pure and applied mathematics [9–15]. It also keeps attracting steady interest in the theoretical physics community, and this over more than fifty years [16–22], with recent applications to statistical physics [21–25], neural networks and complex dynamics [15, 26, 27], string theory [28] and cosmology [29].

Model.— We consider the following energy functional

$$\mathcal{H}[u(\tau)] = \int_0^L d\tau \left[\frac{m^2}{2} u^2(\tau) + V(u(\tau), \tau) + \frac{\kappa}{2} (\partial_\tau u(\tau))^2 \right] \quad (1)$$

where $u(\tau)$, $\tau \in [0, L]$ describes the polymer configuration trajectory (unless stated otherwise we assume the

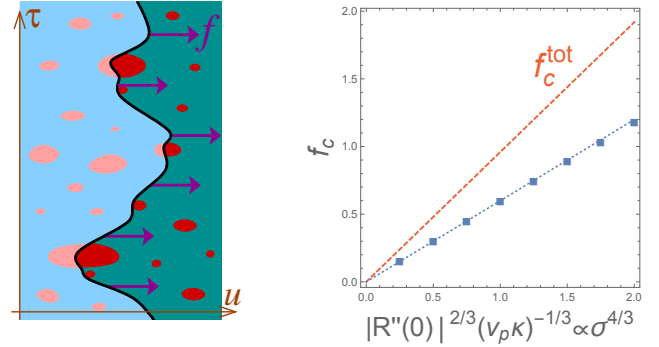


FIG. 1: Left : Elastic line in a disordered medium submitted to a uniform force field. Right : Depinning threshold as a function of the disorder strength $|R''(0)| = \sigma^2$; we compare numerics (squares) and upper bound (5) (dashed line); numerical calculations are performed with averaging over 100 realizations with chains of $K = 512$ monomers.

fixed ends configuration $u(0) = u(L) = 0$), and $\kappa \geq 0$ is the elastic energy coefficient. The random potential $V(u(\tau), \tau)$ is chosen to be Gaussian with zero mean and with the translationally-invariant covariance

$$\langle V(u, \tau) V(u', \tau') \rangle = \delta(\tau - \tau') R(u - u'), \quad (2)$$

where we assume the function $R(u)$ to be at least four times differentiable at $u = 0$. To better define the problem, the polymer is confined inside an harmonic well, of curvature $m^2 \geq 0$, the mass parameter, which flattens the line beyond an infrared length, defined as $L_m := \sqrt{\kappa}/m$. The limit $m \rightarrow 0^+$ is of special interest, as the system becomes critical, with a non trivial roughness exponent, in the $L/L_m \gg 1$ limit [1, 2].

As is well known [1, 2, 30, 31] the zero temperature problem is characterized by the Larkin length L_c , which

gives the order of magnitude of the scale above which metastability (i.e. multiple extrema) sets in. Below that scale, there is typically a unique minimum and the system responds elastically (i.e. linearly) to perturbations, such as an external force $f(\tau)$, i.e. adding $-\int_0^L d\tau f(\tau)u(\tau)$ in (1). In the absence of a mass, standard estimates at weak disorder [1, 2] yield

$$L_c := (\kappa^2/R''''(0))^{1/3}, \quad (3)$$

taken here as a definition. It is also often expressed as $L_c = (\kappa^2 v_p^2 / |R''(0)|)^{1/3}$, where $v_p = (|R''(0)|/R''''(0))^{1/2}$ is a characteristic scale of variation of the random potential. $L_c(m)$ in the presence of a mass, was also estimated: in the simplest (Gaussian replica variational) approximation, it corresponds to the scale beyond which replica symmetry breaking (RSB) occurs [30, 32] while in the more elaborate Functional RG (FRG) treatment [2, 33, 34], it is associated with a cusp (non-analyticity) $R''_{\text{ren}}(0^+) > 0$ which develops in the renormalized force correlator, signaling the appearance of shocks and avalanches [30, 35]. It also provides the so-called Larkin-Ovchinnikov (LO) estimate for the depinning threshold f_c under the action of a uniform force $f(\tau) = f$, as $f_c \sim \kappa v_p / L_c^2 = v_p R''''(0)^{2/3} \kappa^{-1/3}$ [1, 2], such that for $f > f_c$ there are no more barriers to motion [1, 2, 36–43]. These estimates are either dimensional (balancing elastic and pinning energy), or valid in large embedding dimension (RSB) or within an (internal) dimensional expansion (FRG), and at present there are no exact result for the number of minima, or even of equilibria, for a model of a directed polymer in one dimension.

Main results.— Here we provide some exact results for this problem. Namely, for model (1) we show that the mean total number of equilibria grows exponentially at large L , and that the rate r is given in terms of the two length scales L_m and L_c defined above as

$$\langle \mathcal{N}_{\text{tot}} \rangle \sim e^{rL} \quad \text{with} \quad r = \frac{1}{L_m} g\left(\frac{L_m}{L_c}\right), \quad (4)$$

where $g(x)$ is a function calculated below. Although this result is derived for the continuum model (1) and Gaussian disorder, we argue that the function $g(x)$ is universal, for a broader class of models, in the limit of weak disorder $L_c \gg a$, and small mass $L_m \gg a$ (where a is a UV cut-off, such as lattice spacing). We obtain the asymptotic behavior $g(x) \simeq Cx$ at large x , with $C = 0.461 \pm 0.001$, leading to $r = C/L_c$ in the zero mass limit. In the other limit of large mass, $x = L_m/L_c \rightarrow 0$ we find $g(x) \sim x^3 \exp\{-8/(3x^3)\}$, as the equilibria become exponentially rare. Furthermore, the method of calculation developed here is interesting in itself as it reveals connections to other stochastic problems such as, multifractality in quantum 1D localization in a white noise random potential, thermally activated dynamics of a particle near depinning, and probably more, to be explored.

As an application of our result we consider the depinning transition in presence of a uniform applied force f . Predicting the value of f_c is a difficult problem in the theory of pinning. The value f_c of the depinning threshold is identified [37, 38] as the force f beyond which no metastable state survive. Depinning can thus be put in the broader framework of the so-called *topology trivialization* phenomenon, which has been studied recently in the context of spin glasses and other complex dynamical systems [15, 25–27]. Here we calculate $\langle \mathcal{N}_{\text{tot}} \rangle$ and obtain the value, f_c^{tot} at which it drops below unity. As we argue below, this provides an upper bound

$$f_c \leq f_c^{\text{tot}} = \frac{\sqrt{2C}|R''(0)|R''''(0)^{1/3}}{\kappa^{1/3}} = \sqrt{2C} \frac{|R''(0)|^{2/3}}{(v_p \kappa)^{1/3}}, \quad (5)$$

with the constant C given above. The formula for f_c^{tot} is compatible with the LO estimate, and we confirm below from numerical simulations that (5) is a strict inequality.

Counting of equilibria.— To perform the counting of equilibria for the above model it is easier to start with a lattice version of the problem: the continuous variable τ is replaced by a discrete one $i = 1, \dots, K$ with $K = L/a$ (for simplicity we choose units such that the lattice spacing is $a = 1$). The energy and the correlator are then

$$\mathcal{H}\{\mathbf{u}\} = \sum_{i=1}^K \left[\frac{m^2}{2} u_i^2 + V_i(u_i) \right] + \frac{\kappa}{2} \sum_{i=0}^K (u_i - u_{i+1})^2 \quad (6)$$

$$\langle V_i(u) V_j(u') \rangle = \delta_{ij} R(u - u'). \quad (7)$$

Configurations of the polymer are described by vectors of transverse coordinates $\mathbf{u} = (u_1, \dots, u_K)$, with $u_i \in \mathbb{R}$ for $i = 1, 2, \dots, K$ and the convention $u_0 = u_{K+1} = 0$. An equilibrium configuration is found as a solution of the system of K stationarity conditions which can be conveniently written as

$$\partial_{u_i} \mathcal{H}\{\mathbf{u}\} = [(m^2 I - \kappa \Delta) \mathbf{u}]_i + V'_i(u_i) = 0, \quad i = 1, \dots, K \quad (8)$$

where I is the identity matrix and Δ is the *discrete Laplacian matrix* for the underlying one-dimensional lattice, with the only non-zero entries for our choice of boundary conditions being $\Delta_{i,i} = -2$, $i = 1, \dots, K$ and $\Delta_{i,i-1} = \Delta_{i-1,i} = +1$. Note that in the continuum limit such a matrix approximates the standard one-dimensional Laplacian operator $d^2/d\tau^2$ with Dirichlet boundary conditions.

The total number, \mathcal{N}_{tot} , of solutions of such equations is then known to be given by the famous *Kac-Rice* formula, see [9] for a rigorous exposition, and [10] for a review. It can be written as follows

$$\mathcal{N}_{\text{tot}} = \int_{\mathbb{R}^K} d\mathbf{u} \rho(\mathbf{u}) \quad (9)$$

$$\rho(\mathbf{u}) = \left| \det \left(\partial_{u_i, u_j}^2 \mathcal{H} \right) \right| \times \prod_{i=1}^K \delta(\partial_{u_i} \mathcal{H}), \quad (10)$$

where the Hessian is a $K \times K$ matrix given explicitly by

$$\partial_{u_i, u_j}^2 \mathcal{H} = [m^2 + V_i''(u_i)] \delta_{i,j} - \kappa \Delta_{i,j}. \quad (11)$$

The notation (9) allows to express more general quantities such as the total number of solutions $\mathcal{N}_A = \int_A d\mathbf{u} \rho(\mathbf{u})$ such as \mathbf{u} belongs to a subset A of \mathbb{R}^K .

Although (9,10) are valid in an arbitrary disorder realization, we restrict here to the disorder average $\langle \mathcal{N}_{\text{tot}} \rangle$. To perform the disorder average we will use that (a) the potentials $V_i(u)$ and $V_j(u)$ are statistically independent for $i \neq j$ and (b) the variables $V_i'(u)$ are independent of $V_i''(u)$ for any i : indeed, from (7), one has $\langle V_i'(u) V_i''(u) \rangle = R'''(0) = 0$ since we have assumed differentiability ($R(u)$ being obviously an even function). More generally, the property (b) is an important consequence of translational invariance and the Gaussian character of the random function $V_i(u)$ [9]. Moreover, after the average the mod-Hessian factor is obviously independent on \mathbf{u} , and the average of each of the K δ -factors can be done independently over the distribution of the Gaussian variable $V_i'(u)$ with the variance $\langle [V_i'(u)]^2 \rangle = -R''(0)$, which gives $\langle \mathcal{N}_{\text{tot}} \rangle = \langle |\det(\partial_{u_i, u_j}^2 \mathcal{H})| \rangle J(m^2)$ where

$$J(m^2) = \int \frac{d\mathbf{u}}{(2\pi |R''(0)|)^{K/2}} e^{-[(m^2 I - \kappa \Delta)\mathbf{u}]^2 / (2|R''(0)|)}. \quad (12)$$

The Gaussian integral yields the constant Jacobian factor $|\det(m^2 I - \kappa \Delta)|^{-1}$ finally implying that

$$\langle \mathcal{N}_{\text{tot}} \rangle = \frac{\langle |\det(m^2 \delta_{ij} + \kappa [-\Delta_{ij} + U_i \delta_{ij}])| \rangle}{|\det(m^2 \delta_{ij} - \kappa \Delta_{ij})|}, \quad (13)$$

where the averaging goes over the set of i.i.d. mean-zero Gaussian random variables $U_j \equiv V_j''(u_j)/\kappa$ with the correlations $\langle U_i U_j \rangle = 2D \delta_{ij}$, where $D = R''''(0)/(2\kappa^2)$ measures the strength of the disorder in the problem. It is related to the Larkin length at $m = 0$ defined in (3) by $L_c = (2D)^{-1/3}$. The formula (13) can be studied for the discrete model [61], but here we simplify its analysis by considering the continuum limit.

Continuum limit. — The continuous version of the problem is related to the pair of Schrödinger operators $H = -d^2/d\tau^2 + U(\tau)$ and $H_{\text{free}} = -d^2/d\tau^2$, where $U(\tau)$ is a Gaussian white-noise potential with mean zero and covariance $\langle U(\tau)U(\tau') \rangle = 2D \delta(\tau - \tau')$. It is rather apparent that the mean value of the equilibria should be given by a similar mean modulus of the ratio of two functional determinants for the operators $\mathcal{M} = H + m^2/\kappa$ and $\mathcal{M}_{\text{free}} = H_{\text{free}} + m^2/\kappa$ acting on functions vanishing at the two boundaries (Dirichlet boundary conditions).

As is well-known [44, 45], such ratios of functional determinants for Schrödinger operators can be evaluated by so-called Gelfand-Yaglom method as $\det \mathcal{M} / \det \mathcal{M}_{\text{free}} = y(L)/y_{\text{free}}(L)$ in terms of the solutions $y(\tau)$ of the *initial value* problem for the same operators: $\mathcal{M}y = 0$ for

$y(0) = 0$ and $y'(0) = 1$. We have obviously $y_{\text{free}}(\tau) = L_m \sinh(\tau/L_m)$ where $L_m = \sqrt{\kappa}/m$. We deduce

$$\langle \mathcal{N}_{\text{tot}} \rangle = \frac{\langle |y(L)| \rangle}{L_m \sinh(L/L_m)}, \quad (14)$$

where $y(\tau)$ solves the Cauchy initial value problem:

$$-\frac{d^2 y(\tau)}{d\tau^2} + U(\tau) y(\tau) = -\frac{m^2}{\kappa} y(\tau). \quad (15)$$

Cauchy problems related to products of random 2×2 matrices of various types as well as their continuous limits were under active consideration recently, with many important analytical insights, see [46–48] and references therein.

At this point it is appropriate to note that the spectral problem defined by (15) with $y(0) = y(L) = 0$ is the classical problem of one-dimensional localization for the Schrödinger equation with a Gaussian white-noise potential, studied extensively since the seminal work [49], see [50] for further detail. To make contact to notations used for that problem in the literature we introduce $E = -m^2/\kappa = 1/L_m^2$, which plays the role of an effective energy. As is well-known the main qualitative feature of the Cauchy problem (15) is the exponential growth of its solution from chosen initial conditions in every realization of the disorder [50]. It is conventional to define the localisation length as the inverse of the Lyapunov exponent $\gamma_1 = \lim_{L \rightarrow \infty} (1/L) \ln |y(L)| > 0$, which is self-averaging in the limit. However the average $\langle |y(L)| \rangle$ depends crucially on the fluctuations, and is not given by $\exp\{\gamma_1 L\}$. To discuss the role of the fluctuations, as was done in the description of turbulence one uses the multifractal formalism developed by Paladin and Vulpiani [51]. In this approach one introduces the moments $\langle |y(L)|^q \rangle$, with arbitrary positive parameter q . These are known to grow asymptotically as

$$\langle |y(L)|^q \rangle \sim e^{L \Lambda(q)} \quad \text{as } L \rightarrow \infty, \quad (16)$$

where $\Lambda(q)$ is the large deviation rate function, also known as the *generalized* Lyapunov exponent (GLE) [51]. It can be written as a series $\Lambda(q) = \sum_{n=1}^{\infty} (\gamma_n/n!) q^n$ where $\gamma_n L$ is the cumulant of order n of $\ln |y(L)|$ at large L . From Eq. (14), it is related to the rate

$$r = \Lambda(1) - \sqrt{|E|}. \quad (17)$$

For the study of the localisation properties it is natural to consider the so-called *semiclassical regime* of large positive energy, $E \gg D^{2/3}$. In that regime the fluctuations can be considered as Gaussian with subdominant higher cumulants [52], $\Lambda(q) \simeq \gamma_1(q + q^2/2)$, where the Lyapunov exponent can be computed perturbatively, $\gamma_1 \simeq D/(4E^{3/2})$. The property $\gamma_2 \simeq \gamma_1$ is known as “single parameter scaling” [53]. This regime would correspond to $m^2 < 0$ with $|L_m| \ll L_c$ while here, in the

elastic line counting problem, we are interested in the opposite case $m^2 \geq 0$ of negative effective energies $E < 0$. The explicit evaluation of (14) remains an outstanding and non-perturbative problem where non-Gaussian fluctuations dominate (for early works see e.g. [54, 55]).

Stochastic Riccati equation.— A recursive method was developed in [52] allowing to obtain integral representations for the γ_n 's in terms of multiple integrals, but they are quite complicated and not convenient to obtain limiting behaviours. An alternative way of calculating these quantities, suitable for the $E \rightarrow -\infty$ limit (large mass limit for the DP problem), was proposed in [57] for a different model. We adjust this approach here: the transformation $|y(L)| = \exp\{\int_0^L d\tau z(\tau)\}$ relates (15) to the stochastic Riccati equation [56]

$$\frac{dz(\tau)}{d\tau} = -E - z(\tau)^2 + U(\tau) = -\mathcal{U}'(z(\tau)) + U(\tau). \quad (18)$$

For $E < 0$ and $U(\tau) = 0$, $z = \sqrt{|E|}$ is a fixed point. The "noise" $U(\tau)$ generates fluctuations around this point: linearizing the "force", $-E - z^2 \simeq -2\sqrt{|E|}(z - \sqrt{|E|})$, leads to the Ornstein-Uhlenbeck process. The method of [57] provides a systematic analysis of the nonlinearity by perturbation theory around the Ornstein-Uhlenbeck process and leads to express γ_n as a series in powers of D . We get in this way the three first cumulants, corresponding to $\Lambda(q) \simeq \Lambda_{\text{pert}}(q)$ with [61]

$$\frac{\Lambda_{\text{pert}}(q)}{\sqrt{|E|}} = q - \frac{s}{4}q(1-q) - \frac{s^2}{32}q(1-q)(5-4q) + \mathcal{O}(s^3) \quad (19)$$

where $s = D|E|^{-3/2} = (1/2)(L_m/L_c)^3$. Quite remarkably, Eq. (19) shows that the rate r vanishes up to third order in $s \ll 1$ (large mass limit). We conjecture that this remains true at all orders in D , i.e. $\Lambda_{\text{pert}}(q=1) = 0$, which is confirmed by numerics (see below and Fig. 3).

The potential $\mathcal{U}(z) = Ez + z^3/3$ is not confining, thus the noise is not only responsible for small fluctuations around $+\sqrt{|E|}$, characterised by (19), but can also produce large excursions of the process at $\pm\infty$: if the process overcomes the potential barrier at $-\sqrt{|E|}$, it is rapidly driven towards $-\infty$, reinjected at $+\infty$, from which it eventually goes back to $+\sqrt{|E|}$. The rare jumps are separated by time intervals exponentially distributed [58] with exponentially small rate $N(E) \simeq [\sqrt{|E|}/\pi] \exp[-4|E|^{3/2}/(3D)]$ (this is the integrated density of states of the disordered model [49]). This picture suggests to split the process into two parts, describing the small fluctuations around $z = \sqrt{|E|}$ and the rare jumps: $z(\tau) = z_{\text{trapped}}(\tau) + z_{\text{jump}}(\tau)$. Because the two processes involve different time scales, they can be assumed independent and the GLE splits in two parts: $\Lambda(q) \simeq \Lambda_{\text{pert}}(q) + \Lambda_{\text{jump}}(q)$. The contribution of the jumps is estimated by writing $z_{\text{jump}}(\tau) = \sum_n h_n(\tau - \tau_n)$ where $h_n(\tau)$ is a narrow function describing the jump

at time τ_n . Over large scale, $\int_0^L d\tau z_{\text{jump}}(\tau)$ is equivalent to a Compound Poisson process with Lévy exponent $\Lambda_{\text{jump}}(q) \simeq N(E) (\langle e^{qv_n} \rangle - 1)$ where $v_n = \int d\tau h_n(\tau)$ (see [59] and references therein). This simple argument shows that the dependence of the rate in E is mostly controlled by $N(E)$, hence presents the non-analytic behaviour $r = \Lambda(1) - \sqrt{|E|} \simeq \Lambda_{\text{jump}}(1) \sim \exp[-4|E|^{3/2}/(3D)]$. We emphasize that this behaviour is fully controlled by fluctuations, as the non analytic part of the Lyapunov exponent γ_1 is much smaller $\sim \exp[-8|E|^{3/2}/(3D)]$ [61].

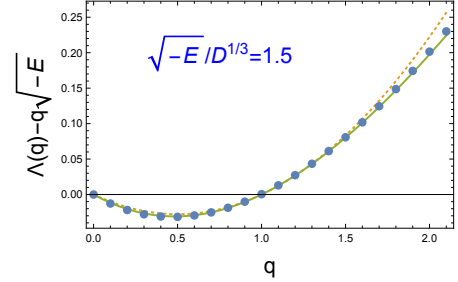


FIG. 2: Generalised Lyapunov exponent obtained numerically (dots). Dotted line is $-(s/4)q(1-q)$ and continuous line (19).

Spectral interpretation.— We now go beyond the previous analysis, which focused only on the large mass limit. The diffusion (18) is characterised by the ("backward") generator \mathcal{G} and its adjoint \mathcal{G}^\dagger ("forward generator")

$$\mathcal{G} = D \frac{d^2}{dz^2} - (E + z^2) \frac{d}{dz}, \quad \mathcal{G}^\dagger = D \frac{d^2}{dz^2} + \frac{d}{dz}(E + z^2) \quad (20)$$

governing the conditional probability density, solution of the Fokker-Plank equation, $\mathcal{P}_\tau(z|z_0) = \langle z | \exp(\tau \mathcal{G}^\dagger) | z_0 \rangle$. The moments (16) are controlled by $\mathcal{O}_q = \mathcal{G}^\dagger + qz$, what provides a spectral interpretation of the GLE: $\Lambda(q)$ is the largest eigenvalue of the non Hermitian operator \mathcal{O}_q . We have studied numerically the spectral problem (cf. [61]). We have first checked the perturbative result (19): we perform a numerical calculation for $D|E|^{-3/2} = 8/27$ and see an excellent agreement (Fig. 2). The rate is plotted in Fig. 3 as a function of $|E|^{3/2}/D$, which confirms the non-analytic behaviour for small disorder $D \rightarrow 0$: the numerical precision allows to determine the pre-exponential factor, yielding [61] $r = \Lambda(1) - \sqrt{|E|} \simeq 0.08(D/|E|) \exp[-4|E|^{3/2}/(3D)]$ (the next eigenvalue of the operator \mathcal{O}_q can be found by an elaborate WKB analysis and presents the same behaviour with a dimensionless factor half [61], which thus provides a lower bound for the rate r). The value of the rate in the opposite $E \rightarrow 0^-$ (i.e. $m^2 \rightarrow 0^+$) limit is also of interest: we have found $r = \Lambda(1) \simeq 0.581 D^{1/3} \simeq 0.461/L_c$ (Fig. 3), a value agreeing with the one deduced from the first 8 cumulants given in Table 1 of [52].

Universality in 1D.— We now comment on the universality of our result Eq. (4). First it is immediate that

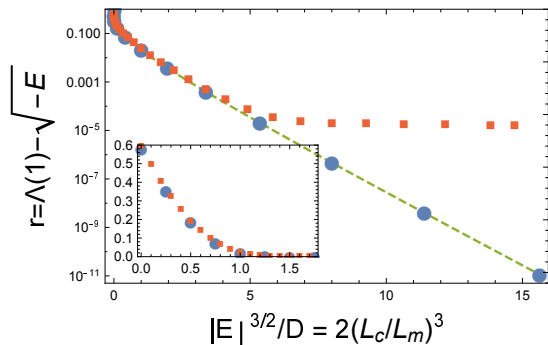


FIG. 3: Rate r obtained numerically by two methods : diagonalization of the discretised \mathcal{O}_q (squares, saturation is a lattice effect [61]) and analysis of the differential equation (blue dots). Dashed line is $0.08D|E|^{-1} \exp[-4|E|^{3/2}/(3D)]$.

(4) can be applied to the discrete model (6) in the limit $L_m, L_c \gg a$ with parameters κ and the same $R(u)$ function (in units such that $a = 1$). In fact, although we will not pursue it here, one can extend the Gelfand-Yaglom method to calculate the ratio of discrete determinants (13) (see Ref. [61]). It is a particular model since it has uncorrelated, Gaussian disorder, and quadratic elastic energy. We expect however that (4) extends to a broader class of DP models (e.g. beyond Gaussian uncorrelated disorder and quadratic elastic energy), with the same universal function, up to two non-universal scales, L_c and L_m (which, in some cases, can be independently measured).

Depinning.— To connect to depinning, we define, in a given sample with a uniform force f , the total number of stable equilibria \mathcal{N}_{st} , i.e. with strictly positive Hessian matrix. We expect that both \mathcal{N}_{st} and \mathcal{N}_{tot} grow exponentially with L , and define [60] the rate functions $\rho_{\text{st}}(f) = \lim_{L \rightarrow \infty} (1/L) \langle \ln \mathcal{N}_{\text{st}} \rangle$, and similarly $\rho_{\text{tot}}(f) = \lim_{L \rightarrow \infty} (1/L) \langle \ln \mathcal{N}_{\text{tot}} \rangle$. These two rates are expected to vanish at the *same* value of the applied force f , which defines the depinning threshold f_c . This is because (i) the no-crossing rule (or Middleton theorems), known to hold for interface depinning, implies that in any given sample the last equilibrium which disappears upon increasing f is a stable equilibrium, (ii) the sample-dependent threshold force at which this happens has fluctuations decaying to zero at large L . Here, instead, we have introduced the rates $r_{\text{tot}}(f) = \lim_{L \rightarrow \infty} (1/L) \ln \langle \mathcal{N}_{\text{tot}} \rangle$ and $r_{\text{st}}(f) = \lim_{L \rightarrow \infty} (1/L) \ln \langle \mathcal{N}_{\text{st}} \rangle$, and have defined f_c^{tot} and f_c^{st} as the forces for which these rates vanish (i.e. $\langle \mathcal{N}_{\text{tot,st}} \rangle \sim 1$ for $f = f_c^{\text{tot,st}}$). From $\mathcal{N}_{\text{st}} \leq \mathcal{N}_{\text{tot}}$ and the convexity of the logarithm we obtain

$$f_c \leq f_c^{\text{st}} \leq f_c^{\text{tot}}. \quad (21)$$

The calculations of the exact f -dependence of $r_{\text{tot}}(f)$ and $r_{\text{st}}(f)$ and of the exact value f_c^{tot} displayed in (5) are detailed in [61] (we also argue why obtaining f_c^{st} is much

more difficult and goes beyond this paper).

We computed f_c numerically for a discrete elastic chain of K monomers using the algorithm developed in [37] : the presence of a stationary state is detected for a given force f . The search is ended when no stationary state are found and all monomers have moved forward of at least two periods of the disorder. The critical force of the sample, defined as the maximal force at which a stationary state still exists, is found by bisection. The energy of the chain is given by Eq. (6) with $m^2 = 0$ and $\kappa = 1$, the random force acting on each monomer is set to be $V_i(u) = \xi_1 \cos u + \xi_2 \sin u$ with ξ_1, ξ_2 independent Gaussian numbers of zero mean and variance σ^2 . Thus the disorder is periodic and characterised by translational invariant covariance with $-R''(0) = R'''(0) = \sigma^2$. Our results (Fig. 1) show that $f_c^{\text{tot}}/f_c = 1.64 \pm 0.02$. Boundary conditions, both for the numerical calculation of f_c and the analytical calculation of f_c^{tot} are discussed in [61].

The connections between the pinning and localization theories established in this work extend to interfaces of higher internal dimension d , or replacing the Laplacian by a more general elastic matrix (short range or long range) $-\Delta_{ij} \rightarrow K_{ij}$. Since the formula (13) can be naturally generalized, we obtain a formula for f_c^{tot} in the general case, in terms of a determinant, $\langle |\det(K_{ij} + U_i \delta_{ij})| \rangle$, which however, remains to be evaluated in arbitrary d , see details in [61].

Conclusion.— Using the Kac-Rice formula, we have shown that the number of equilibria of an elastic line in a random potential grows exponentially with its length, and developed a theory to calculate the rate of growth. The latter is described by a universal function of the disorder strength for which we obtained analytical and numerical results. This allowed us to obtain non-trivial information about pinning and an exact upper bound on the depinning threshold. The connection discussed here with the generalized Lyapunov exponents of a localization problem opens a bridge between pinning theory and localization theory that should inspire further works.

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Exponential number of equilibria and depinning threshold for a directed polymer in a random potential – SUPPLEMENTAL MATERIAL

A. BOUNDARY CONDITIONS AND DETERMINANTS

In this section, we discuss several formulae for the determinant $\det(H - E)$ of the Schrödinger operator to justify and make more precise the formula of the main text and of Section B below.

Consider the discrete Hamiltonian which appears in the text, $H_{i,j} = -\Delta_{i,j} + U_i\delta_{i,j}$ and the associated Schrödinger equation

$$\sum_j H_{i,j}\psi_j = -\psi_{i+1} + (2 + U_i)\psi_i - \psi_{i-1} = E\psi_i \quad (22)$$

for $i \in \{1, \dots, K\}$; boundary conditions set the values for ψ_0 and ψ_{K+1} .

A. 1. Boundary conditions

There are three natural boundary conditions for the elastic line problem studied in this paper. We detail them here in the discrete and continuum settings.

- The pinned boundary conditions correspond to setting $\psi_0 = \psi_{K+1} = 0$ in (22) ($u_0 = u_{K+1} = 0$ with the notations of the paper). The associated $K \times K$ Laplacian matrix $\Delta = \Delta_{\text{Dir}}$ then reads :

$$\Delta_{\text{Dir}} = \begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & -2 & 1 \\ 0 & \dots & 0 & 1 & -2 \end{pmatrix} \quad (23)$$

In the continuum limit it corresponds to the usual Laplacian with Dirichlet boundary conditions $\psi(0) = \psi(L) = 0$.

- The free boundary conditions : set $\psi_0 = \psi_1$ and $\psi_K = \psi_{K+1}$ in (22), i.e. the associated Laplacian $\Delta = \Delta_{\text{Neu}}$ is

$$\Delta_{\text{Neu}} = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & -2 & 1 \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix} \quad (24)$$

In the continuum limit it corresponds to usual Laplacian with Neumann boundary conditions $\psi'(0) = \psi'(L) = 0$.

- The periodic boundary conditions, $\psi_0 = \psi_K$, $\psi_{K+1} = \psi_1$, with the associated Laplacian $\Delta = \Delta_{\text{Per}}$ given by

$$\Delta_{\text{Per}} = \begin{pmatrix} -2 & 1 & 0 & \dots & 1 \\ 1 & -2 & 1 & 0 & \dots \\ 0 & 1 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & -2 & 1 \\ 1 & \dots & 0 & 1 & -2 \end{pmatrix} \quad (25)$$

In the continuum limit this corresponds to usual Laplacian for $\psi(0) = \psi(L)$ and $\psi'(0) = \psi'(L)$.

A. 2. Determinants

We now discuss separately the formulae for the determinant $\det(H - E)$ for the three types of boundary conditions.

Pinned boundary conditions (Dirichlet).— We denote by $y_i(E)$ the solution of the initial value problem with

$$y_0 = 0 \quad \text{and} \quad y_1 = 1. \quad (26)$$

Then we see by recursion that $y_2(E) = 2 + U_1 - E$, $y_3(E) = (2 + U_2 - E)(2 + U_1 - E) - 1$, etc. i.e. $y_{j+1}(E)$ is a polynomial of degree j in E with higher degree term $(-E)^j$. The spectrum is given by the quantization condition

$$y_{K+1}(E) = 0. \quad (27)$$

Using these remarks we easily see that

$$\det(H - E) = \prod_{\alpha=1}^K (E_{\alpha} - E) = y_{K+1}(E) \quad (28)$$

where the E_{α} are the eigenvalues of H . This can now be used for the discrete polymer model, formula (13) in the text, with the correspondence $E = -m^2/\kappa$. As a simple illustration consider the free case with $U_n = -2$. We deduce $y_n = \sin qn / \sin q$ where $E = -2 \cos q$. As a consequence the Dirichlet determinant is $\det(H - E) =$

$\sin q(K+1)/\sin q$, corresponding to the spectrum $q_n = n\pi/(K+1)$ with $n = 1, \dots, K$.

Denoting $y(\tau)$ the solution of $(H - E)y(\tau) = 0$ with initial conditions

$$y(0) = 0 \quad \text{and} \quad y'(0) = 1, \quad (29)$$

we see that Eq. (28) is obviously the discrete version of the Gelfand-Yaglom formula

$$\det(H - E) = 2y(L). \quad (30)$$

up to a E independent multiplicative factor which depends on the ultraviolet regularization (the factor chosen here and formulae given below correspond to zeta regularization [T10, HKT12]). Let us illustrate the formula in the free case where $H = -\partial_\tau^2$: setting $\gamma = -E$ for convenience for the following, we find $y(\tau) = \sinh(\sqrt{\gamma}\tau)/\sqrt{\gamma}$ thus $\det(\gamma - \partial_\tau^2) = 2 \sinh(\sqrt{\gamma}L)/\sqrt{\gamma}$. Rewriting the hyperbolic function as an infinite product

$$\det(\gamma - \partial_\tau^2) = 2L \prod_{n=1}^{\infty} \left(1 + \frac{\gamma}{q_n^2}\right), \quad q_n = \frac{n\pi}{L} \quad (31)$$

we recognize the eigenvalues of the Laplacian. In this case we recall the eigenfunctions, $-\partial_\tau^2 \psi_n(\tau) = q_n^2 \psi_n(\tau)$,

$$\psi_n(\tau) = \sqrt{\frac{2}{L}} \sin(q_n \tau) \quad \text{with } n \in \mathbb{N}^*. \quad (32)$$

Free boundary conditions (Neumann).— $\psi_0 = \psi_1$ and $\psi_K = \psi_{K+1}$. We now denote by $x_i(E)$ the solution of the initial value problem with

$$x_0 = x_1 = 1 \quad (33)$$

Now the quantization condition is

$$x_{K+1}(E) = x_K(E) \quad (34)$$

Similar argument as above gives

$$\det(H - E) = x_{K+1}(E) - x_K(E). \quad (35)$$

In the continuum limit, we consider the solution $x(\tau)$ of the differential equation $(H - E)x(\tau) = 0$, where $H = -\frac{d^2}{d\tau^2} + U(\tau)$, with initial conditions

$$x(0) = 1 \quad \text{and} \quad x'(0) = 0 \quad (36)$$

The determinant is then given by

$$\det(H - E) = 2x'(L) \quad (37)$$

As in the text, a regularization independent formula is obtained by forming ratio of two such determinants. We illustrate the formula in the free case: setting again $E = -\gamma$ we find $x(\tau) = \cosh(\sqrt{\gamma}\tau)/\sqrt{\gamma}$, thus $\det(\gamma - \partial_\tau^2) = 2\sqrt{\gamma} \sinh(\sqrt{\gamma}L)$, which can also be expressed in terms of the spectrum of eigenvalues $\{q_n^2\}_{n=0,1,\dots}$ of the Laplacian:

$$\begin{cases} \psi_0(\tau) = \frac{1}{\sqrt{L}} \\ \psi_n(\tau) = \sqrt{\frac{2}{L}} \cos(q_n \tau) \end{cases}, \quad q_n = \frac{n\pi}{L} \quad \text{with } n \in \mathbb{N} \quad (38)$$

Periodic boundary conditions.— For the sake of completeness, let us add a phase and now write the periodic boundary conditions as

$$\psi_K = \psi_0 e^{i\theta} \quad \text{and} \quad \psi_{K+1} = \psi_1 e^{i\theta}, \quad (39)$$

(this corresponds to a ring pierced by a magnetic flux). The spectral analysis can be formulated conveniently by introducing two specific solutions $y_i(E)$ and $\tilde{y}_i(E)$ for two different initial value problems corresponding to

$$\begin{cases} y_0 = 0 \\ y_1 = 1 \end{cases} \quad \text{and} \quad \begin{cases} \tilde{y}_{K+1} = 0 \\ \tilde{y}_K = 1 \end{cases} \quad (40)$$

We have pointed out above that $y_n(E)$ is a polynomial of degree $n-1$ in E ; similarly $\tilde{y}_n(E)$ a polynomial of degree $K-n-1$. The Wronskian $W_j = y_j \tilde{y}_{j+1} - y_{j+1} \tilde{y}_j$ of the two solutions is constant (starting from (22), one finds $W_{j+1} = W_j$), and is simply denoted $W = -\tilde{y}_0 = -y_{K+1}$ (which is the Dirichlet determinant).

The solution of the spectral problem can then be decomposed as $\psi_j = a y_j + b \tilde{y}_j$, where a and b are two constants. Imposing the boundary conditions (39) leads to the quantization equation

$$\det \begin{pmatrix} y_K & 1 - \tilde{y}_0 e^{i\theta} \\ e^{i\theta} - y_{K+1} & \tilde{y}_1 e^{i\theta} \end{pmatrix} = 0. \quad (41)$$

Expanding this equation gives

$$\det(H - E) = \frac{1}{\tilde{y}_0} + \tilde{y}_0 - \frac{\tilde{y}_1 y_K}{\tilde{y}_0} - 2 \cos \theta. \quad (42)$$

As an example of application, we consider the free case ($U_n = -2$) where $y_n = \sin qn/\sin q = \tilde{y}_{K+1-n}$ for energy $E = -2 \cos q$. Some algebra gives $\det(H - E) = 2(\cos qK - \cos \theta) = 2(T_K(-E/2) - \cos \theta)$, where $T_K(x)$ is a Chebyshev polynomial.

We can proceed along the same lines in the continuum: we consider the two solutions $y(\tau)$ and $\tilde{y}(\tau)$ of the differential equation with initial conditions

$$\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases} \quad \text{and} \quad \begin{cases} \tilde{y}(L) = 0 \\ \tilde{y}'(L) = -1 \end{cases} \quad (43)$$

The determinant is then given by

$$\det(H - E) = y'(L) - \tilde{y}'(0) - 2 \cos \theta \quad (44)$$

In the free case we have simply $\tilde{y}(\tau) = y(L - \tau)$ and we find $\det(\gamma - \partial_\tau^2) = 2[\cosh(\sqrt{\gamma}L) - \cos \theta]$, which can again be expanded over the spectrum of eigenvalues

$$\det(\gamma - \partial_\tau^2) = 4 \sin^2(\theta/2) \prod_{n \in \mathbb{Z}} \left(1 + \frac{\gamma}{(q_n - \theta/L)^2}\right) \quad (45)$$

We recall also the related eigenfunctions for completeness

$$\psi_n(\tau) = \frac{1}{\sqrt{L}} e^{iq_n \tau}, \quad q_n = \frac{2n\pi}{L} \quad \text{with } n \in \mathbb{Z} \quad (46)$$

For references on functional determinants, cf. Refs. [D00, CDT05, T10, HKT12] and references therein.

B. CALCULATION OF THE UPPER BOUND FOR THE DEPINNING THRESHOLD: DETAILS

In this section we study the elastic line in presence of an applied force f . We discuss the definition of the depinning threshold and how to obtain upper bounds for it. In particular, we give the detailed derivation of the formula for f_c^{tot} displayed in the main text. We start with $d = 1$ and briefly describe the extension to $d > 1$ at the end.

The depinning threshold force f_c , is usually defined as the force such that for $f = f_c$ barriers in the energy landscape disappear. For the interface problem it also corresponds to the force at which stable equilibria, i.e. metastable states, disappear (see [KBFR13] and references therein). As discussed in the main text, it should thus be identified with the value of f at which the typical number of metastable states, i.e. $\rho_{\text{st}}(f) = \lim_{L \rightarrow \infty} (1/L) \langle \ln \mathcal{N}_{\text{st}} \rangle$, vanishes. A natural first upper bound for f_c could thus be obtained, in principle, as the value $f = f_c^{\text{st}}$, at which the mean number of metastable states $\langle \mathcal{N}_{\text{st}} \rangle \sim 1$. Although one can write a formula for this quantity, which we display below, its exact evaluation remains an interesting challenge.

Hence we focus here on the mean total number of equilibria, $\langle \mathcal{N}_{\text{tot}} \rangle$ and obtain an exact formula for f_c^{tot} , the force at which $\langle \mathcal{N}_{\text{tot}} \rangle \sim 1$, providing another, less stringent, upper bound for f_c (see main text). To define properly f_c^{tot} , as already the case for f_c , one needs to specify further the boundary conditions, and to adapt slightly the setting and the method described in the main text.

Toy model ($d = 0$)

For simplicity let us start with a toy model of a single particle, of energy $\mathcal{H}(u) = m^2 u^2/2 + V(u) - fu$ (i.e. our discrete model (6) with $K = 1$ and $\kappa = 0$), in presence of an external force f . Let us calculate the number of equilibria, denoted here \mathcal{N}_w , weighted by a function $\phi(u)$ of width w :

$$\mathcal{N}_w := \int du \phi(u) \rho(u) \quad (47)$$

$$\rho(u) = |m^2 + V''(u)| \delta(m^2 u - f + V'(u)) \quad (48)$$

where $\rho(u)$ is the mean density of equilibria at point u . From the text, its mean value is given by

$$\langle \mathcal{N}_w \rangle = G(m^2) J_w(m^2) \quad (49)$$

where

$$G(m^2) := \langle |m^2 + V''(u)| \rangle \quad (50)$$

$$= \sqrt{\frac{2R''''(0)}{\pi}} e^{-m^4/(2R''''(0))} + m^2 \operatorname{erf} \left(\frac{m^2}{\sqrt{2R''''(0)}} \right)$$

with $G(0) = \sqrt{2R''''(0)}/\pi$, and

$$J_w(m^2) := \langle \delta(m^2 u - f + V'(u)) \rangle \quad (51)$$

$$= \int \frac{du \phi(u)}{\sqrt{2\pi|R''(0)|}} e^{-\frac{1}{2|R''(0)|}(m^2 u - f)^2} \quad (52)$$

Now note that if we choose $\phi(u) = 1$, then $J_\phi(m^2) = 1/m^2$ independently of f , and we obtain the mean total number of equilibria as $\langle \mathcal{N}_{\text{tot}} \rangle = G(m^2)/m^2$. For $m^2 > 0$ it makes sense that it is independent of f , since the position of the center of the harmonic confining well is just shifted by f/m^2 . In the case most relevant for depinning, $m^2 \rightarrow 0^+$, this number diverges (as the particle can explore the whole real axis) and we cannot conclude.

To better define the problem (in the limit $m^2 \rightarrow 0^+$) we will now count the mean number of equilibria in a finite region of space u , of width w . In practice this is how a depinning force is defined, e.g. by restricting the particle to move on a cylinder of perimeter w . Here we could restrict $u \in [-w/2, w/2]$, but for calculational simplicity let us choose instead $\phi(u) = e^{-u^2/(2w^2)}$, a choice indicated by the subscript w . This gives

$$J_w(m^2) = \int \frac{du}{\sqrt{2\pi|R''(0)|}} e^{-\frac{1}{2|R''(0)|}(m^2 u - f)^2 - u^2/(2w^2)}$$

$$= \frac{e^{-f^2/[2(m^4 w^2 + |R''(0)|)]}}{\sqrt{m^4 + |R''(0)|}/w^2} \quad (53)$$

In the limit $m^2 \rightarrow 0^+$ we find

$$\langle \mathcal{N}_w \rangle = w \sqrt{\frac{2R''''(0)}{\pi|R''(0)|}} e^{-f^2/(2|R''(0)|)} \quad (54)$$

Hence we find that the mean number drops below unity when $f > f_c^{\text{tot}}$ with

$$f_c^{\text{tot}} = \sqrt{2|R''(0)| \left(\ln w + \frac{1}{2} \ln \frac{2R''''(0)}{\pi|R''(0)|} \right)} \quad (55)$$

This is the force such that the mean total number of equilibria drop below unity in the interval of width $\sim w$, in the u -space. Note that the effect of the mass m can be neglected as long as $w \ll \sqrt{|R''(0)|}/m^2$, and that in $d = 0$ the number of equilibria is at most of order twice the number of metastable states. The square-root logarithmic dependence in the width w is thus not surprising for the model of a particle as it originates from rare large barriers in the tail of the Gaussian, and is compatible with calculations of the depinning threshold in Ref. [LW09a] on related models. Due to this effect there is no true finite depinning threshold force for the particle for $w \rightarrow +\infty$ (except for bounded disorder). We now turn to the elastic line, which does admit a well-defined depinning threshold force in the thermodynamic limit, even for Gaussian disorder.

Elastic line ($d = 1$) – discrete model

Let us now generalise to the DP model, using the restriction $\phi(\mathbf{u}) = e^{-\frac{\mathbf{u}^2}{2w^2}}$. We need to calculate

$$J_w(m^2) = \int \frac{d\mathbf{u}}{(2\pi|R''(0)|)^{K/2}} e^{-\frac{\mathbf{u}^2}{2w^2}} \times \exp \left\{ -\frac{[(m^2 \mathbf{1}_K - \kappa \Delta) \mathbf{u} - \mathbf{f}]^2}{2|R''(0)|} \right\} \quad (56)$$

where we have added a term $-\sum_{i=1}^K f_i u_i = -\mathbf{f} \cdot \mathbf{u}$ to the energy (6). $\mathbf{1}_K$ is the identity matrix of size K . Performing the integrals, this leads to a modification of (13)

$$\langle \mathcal{N}_w \rangle = \frac{\langle |\det(m^2 \mathbf{1}_K - \kappa \Delta + \kappa U)| \rangle}{\sqrt{\det((m^2 \mathbf{1}_K - \kappa \Delta)^2 + \frac{|R''(0)|}{w^2} \mathbf{1}_K)}} \quad (57)$$

$$\times \exp \left[-\frac{1}{2w^2} \mathbf{f}^T \left[(m^2 \mathbf{1}_K - \kappa \Delta)^2 + \frac{|R''(0)|}{w^2} \mathbf{1}_K \right]^{-1} \mathbf{f} \right]$$

where U denotes the diagonal matrix with elements $U_i \delta_{ij}$. Using $\det[(a \mathbf{1}_K - \Delta)^2 + b^2 \mathbf{1}_K] = \det[a \mathbf{1}_K - \Delta + i b \mathbf{1}_K] \det[a \mathbf{1}_K - \Delta - i b \mathbf{1}_K]$, the expression can be further simplified as

$$\langle \mathcal{N}_w \rangle = \frac{\langle |\det(L_m^{-2} \mathbf{1}_K - \Delta + U)| \rangle}{|\det((L_m^{-2} + i L_w^{-2}) \mathbf{1}_K - \Delta)|} \quad (58)$$

$$\times \exp \left[-\frac{1}{2(w\kappa)^2} \mathbf{f}^T [(L_m^{-2} \mathbf{1}_K - \Delta)^2 + L_w^{-4} \mathbf{1}_K]^{-1} \mathbf{f} \right]$$

We have introduced the two length scales

$$L_m := \frac{\sqrt{\kappa}}{m} \quad \text{and} \quad L_w := \frac{\sqrt{w\kappa}}{|R''(0)|^{1/4}} = \sqrt{\frac{w}{v_p L_c^{1/2}}} L_c. \quad (59)$$

The formula (58) is valid for arbitrary number of monomers K and general initial conditions, i.e. for all three types studied in Section A. The criterion $\langle \mathcal{N}_w \rangle \sim 1$ then allows to obtain a bound on the depinning threshold in the discrete setting (see below for an application).

Elastic line ($d = 1$) – continuous model

Taking the continuum limit is now straightforward. In presence of an external force, adding the term $-\int_0^L d\tau f(\tau) u(\tau)$ to the energy functional (1). To restrict to a finite region in u -space we similarly introduce a functional $\phi[u] = \exp \left\{ -(1/2w^2) \int_0^L d\tau u(\tau)^2 \right\}$. Note that the dimension of w^2 is now $[u]^2[L]$. The extension of (58) is

$$\langle \mathcal{N}_w \rangle = \frac{\langle |\det(L_m^{-2} - \partial_\tau^2 + U(\tau))| \rangle}{|\det(L_m^{-2} + i L_w^{-2} - \partial_\tau^2)|} \exp \left\{ -\frac{1}{2(w\kappa)^2} \int_0^L d\tau d\tau' f(\tau) \langle \tau | \frac{1}{(L_m^{-2} - \partial_\tau^2)^2 + L_w^{-4}} | \tau' \rangle f(\tau') \right\} \quad (60)$$

where we have used a quantum mechanical notation for the propagator. Decomposing the force over the eigenmodes $\psi_n(\tau)$ of the Laplace operator (given in section A for the various boundary conditions) with components $\tilde{f}_n = \int_0^L d\tau \psi_n^*(\tau) f(\tau)$ we rewrite more explicitly

$$\langle \mathcal{N}_w \rangle = \frac{\langle |\det(L_m^{-2} - \partial_\tau^2 + U(\tau))| \rangle}{|\det(L_m^{-2} + i L_w^{-2} - \partial_\tau^2)|} \exp \left\{ -\frac{1}{2(w\kappa)^2} \sum_n \frac{|\tilde{f}_n|^2}{(L_m^{-2} + q_n^2)^2 + L_w^{-4}} \right\} \quad (61)$$

where $-\partial_\tau^2 \psi_n(\tau) = q_n^2 \psi_n(\tau)$ (see section A).

The determinant in the denominator is easily obtained for various boundary conditions, Dirichlet, Neumann or

periodic (cf. Section A) [T10] :

$$\det(\gamma - \partial_\tau^2) = \begin{cases} \frac{2}{\sqrt{\gamma}} \sinh \sqrt{\gamma} L & (\text{Dir/Dir}) \\ 2 \cosh \sqrt{\gamma} L & (\text{Dir/Neu}) \\ 2\sqrt{\gamma} \sinh \sqrt{\gamma} L & (\text{Neu/Neu}) \\ 2(\cosh \sqrt{\gamma} L - 1) & (\text{periodic}) \end{cases} \quad (62)$$

We stress that the leading behaviour of the determinant is independent of the boundary conditions in the large L limit $\det(\gamma - \partial_\tau^2) \sim \exp(\sqrt{\gamma}L)$.

Neumann or periodic boundary conditions

We consider a constant force $f(\tau) = f$. The double integral in the exponential of Eq. (60) is most easily computed for Neumann and periodic boundary conditions, as the Laplacian possesses a zero mode $\psi_0(\tau) = 1/\sqrt{L}$ in this case. Using (61) with $\tilde{f}_n = f\sqrt{L}\delta_{n,0}$ shows that (minus) the argument of the exponential in (61) is

$$\frac{|\tilde{f}_0|^2}{2(w\kappa)^2(L_m^{-4} + L_w^{-4})} = \frac{f^2 L}{2(m^4 w^2 + |R''(0)|)} \quad (63)$$

The modulus of the determinant in the denominator of (60,61) is $|\det(L_m^{-2} + iL_w^{-2} - \partial_\tau^2)| \sim \exp[L \operatorname{Re}\sqrt{1/L_m^2 + i/L_w^2}]$ when $L \gg L_m, L_w$. Finally we deduce

$$\begin{aligned} r_{w,m}(f) &:= \lim_{L \rightarrow \infty} \frac{\ln \langle \mathcal{N}_w \rangle}{L} \\ &= \Lambda(1) - \operatorname{Re} \sqrt{\frac{1}{L_m^2} + \frac{i}{L_w^2}} - \frac{f^2}{2(m^4 w^2 + |R''(0)|)} \end{aligned} \quad (64)$$

where $\Lambda(1) = \lim_{L \rightarrow \infty} (1/L) \ln \langle |\det(L_m^{-2} - \partial_\tau^2 + U(\tau))| \rangle$ was studied in the main text. This form is now appropriate to consider first the limit $m^2 \rightarrow 0$ at fixed w , and second the limit $w \rightarrow \infty$ (which clearly do not commute). The first limit leads to $r_w(f) := \lim_{m^2 \rightarrow 0} r_{w,m}(f)$, with

$$r_w(f) = \frac{C}{L_c} - \frac{1}{\sqrt{2}L_w} - \frac{f^2}{2|R''(0)|} \quad (65)$$

which is valid for $L_m, L \gg L_w, L_c$. Taking now the limit $w \rightarrow \infty$, i.e. $L_w \gg L_c$, we obtain $r_{\text{tot}}(f) := \lim_{w \rightarrow \infty} r_w(f)$, with

$$r_{\text{tot}}(f) = \frac{C}{L_c} - \frac{f^2}{2|R''(0)|}. \quad (66)$$

We define the threshold $f_c^{\text{tot}} = \sqrt{2r|R''(0)|}$ as the value of f for which this rate vanishes (i.e. $\langle \mathcal{N}_{\text{tot}} \rangle \sim 1$). We get $f_c^{\text{tot}} = \sqrt{2C|R''(0)|/L_c}$, i.e.

$$f_c^{\text{tot}} = \sqrt{2C} \frac{v_p \kappa}{L_c^2} = \frac{\sqrt{2C|R''(0)|R'''(0)^{1/3}}}{\kappa^{1/3}} \quad (67)$$

as displayed in the text. It is interesting to note that it has a similar order of magnitude as the Larkin-Ovchinnikov (LO) formula [BFGV94, L10]

$$f_c^{\text{LO}} = \frac{|R''(0)|^{2/3}}{(v_p \kappa)^{1/3}} \quad (68)$$

with $R'''(0) = |R''(0)|/v_p^2$, which, however is only an order of magnitude estimate. As discussed in the text, our result (67) is an exact upper bound for the true f_c in the continuous model, or the discrete one in the limit $L_c \gg a$.

Dirichlet boundary conditions

Let us comment on the dependence in the boundary conditions. We can also perform the calculation for an elastic line pinned at its two ends (Dirichlet boundary conditions) [19]. The Fourier coefficients are

$$\tilde{f}_n = \frac{2f\sqrt{2L}}{n\pi}, \quad n = 2p + 1 \quad (69)$$

$$= 0, \quad n = 2p + 2 \quad (70)$$

for $p = 0, 1, 2, \dots$. Performing the sum over the modes, $q_n = n\pi/L$ with $n \in \mathbb{N}^*$, one finds that (minus) the argument of the exponential in (61) becomes

$$\frac{Lf^2}{2|R''(0)|} \sum_{n \text{ odd}} \frac{8}{(n\pi)^2} \frac{1}{1 + (L_w/L)^4 [(n\pi)^2 + (L/L_m)^2]^2}. \quad (71)$$

In the limit $L_m/L \rightarrow \infty$ (vanishing mass) and $L/L_w \rightarrow \infty$, we find

$$\frac{Lf^2}{2|R''(0)|} \sum_{n \text{ odd}} \frac{8}{(n\pi)^2} = \frac{Lf^2}{2|R''(0)|} \quad (72)$$

leading to a value for the second term in (66) *identical* to the case of periodic and Neumann boundary conditions (despite the summation over an infinite set of modes). In addition, we also expect that $\Lambda(1)$, the first term in (66) is independent of the boundary conditions. For instance, in the framework of the Riccati equation (18) defined in the text, the initial condition is $z(0) = -\infty$ for Dirichlet, and $z(0) = 0$ for Neumann boundary conditions. However, it is known that the rate of growth, i.e. the GLE, does not depend on the initial condition. Hence $\Lambda(1)$, and the rate $r_{\text{tot}}(f)$ in (66) is the same for all three types of boundary conditions, leading to the same value of f_c^{tot} .

Concluding remarks

Let us conclude by some remarks on the validity of the present calculation. Our starting formula (61) for the mean number of equilibria $\langle \mathcal{N}_w \rangle$ is exact for arbitrary m, w, L . Our strategy to obtain a robust value for f_c^{tot} (independent of details) was to consider $m \rightarrow 0$ first, then $L \rightarrow +\infty$ and only at the end $w \rightarrow +\infty$, so that the condition $m^4 w^2 \ll |R''(0)|$ is always satisfied. With that procedure we found that the criterion $\langle \mathcal{N}_w \rangle \sim 1$ identifies f_c^{tot} unambiguously, independently of any further details

of the boundary conditions: in particular it is independent of the type and range of correlations of the disorder, whether random field, random bond, or random periodic. Hence it is an upper bound on f_c in *any* type of disorder.

It is useful to recall that a similar robustness of the depinning threshold f_c with respect to boundary conditions was observed in Ref. [KBFR13] (and previous works cited there). There f_c was studied for an elastic line on a cylinder of width $W = k v_p (L/L_c)^\zeta$, where ζ is the roughness exponent at depinning (and $m = 0$). The latter is measured from the roughness of the last metastable configuration encountered as f is increased towards f_c . The value of f_c was found independent of the aspect ratio k of the cylinder when both L and W become large. Only finite size corrections, which are subdominant, depend on the aspect ratio and other details. These subdominant sample to sample fluctuations of the depinning threshold force were also studied in Ref. [FLW06, LW09a].

Let us now comment on the respective order of limits of large L and large w . In the present calculation, a finite value of w means that any equilibrium configuration which extends beyond a width $W_L = w/\sqrt{L}$ is not counted in \mathcal{N}_w . If we want the upper bound argument to hold, we only need that this width be larger (or of the same order) than the typical width at depinning, i.e. $W_L \gg v_p (L/L_c)^\zeta$, equivalent to $w \gg v_p L_c^{1/2} (L/L_c)^{\zeta+\frac{1}{2}}$. On the other hand, our intermediate result (65) for the finite w rate, $r_w(f)$, requires $L/L_w \gg 1$. From the definition (59) of L_w , that is equivalent to $w \ll v_p L_c^{1/2} (L/L_c)^2$. Hence whenever $\zeta < 3/2$, which is the case for most classes of disorder, both conditions can be met simultaneously: one can safely use formula (65) and the term $1/L_w$ is then negligible in the thermodynamic limit $W \sim L^\zeta \rightarrow \infty$, leading to (66) and (67). For the case of periodic depinning studied numerically in the main text, $\zeta = 3/2$ (the roughness of the Larkin model). In that case $W \sim L^\zeta$ is equivalent to $L \sim L_w$ and one cannot use the second term in the asymptotic rate formula (65). Instead one must replace it by $-\frac{1}{L} \ln |\det(L_w^{-2} - \partial_\tau^2)|$. An estimate of this quantity however, shows that it is

again negligible in the thermodynamics limit.

Interface model on arbitrary graph and dimension d

As mentionned in the text our work establishes a connection between the pinning and localization theories. In the 1D continuum elastic line model it is best illustrated by rewriting the threshold force f_c^{tot} (67) as

$$\frac{(f_c^{\text{tot}})^2}{2|R''(0)|} = \lim_{L \rightarrow \infty} \frac{\ln \langle |\det(-\partial_\tau^2 + U(\tau))| \rangle}{L} \quad (73)$$

From (58) and following the same steps, a similar formula can be written for the discrete model (6). It can in fact be generalized further, to an interface u_i , $i \in \mathbb{Z}^d$ of internal dimension d , with an arbitrary elastic matrix $-\Delta_{ij} \rightarrow K_{ij}$, for instance $K_q \sim q^a$ in Fourier, with $a = 2$ ($a < 2$) for short-range (long-range) elasticity - and an on-site Gaussian random potential correlated as in (6). The same method leads to a general formula for the threshold force f_c^{tot} , *which is again an upper bound for the depinning threshold force*, $f_c \leq f_c^{\text{tot}}$ with

$$\frac{(f_c^{\text{tot}})^2}{2|R''(0)|} = \lim_{\Omega \rightarrow \infty} \frac{\ln \langle |\det(K_{ij} + U_i \delta_{ij})| \rangle}{\Omega} \quad (74)$$

where Ω is the volume of the system and U_i a Gaussian random potential with correlator $\langle U_i U_j \rangle = (R''''(0)/\kappa^2) \delta_{ij}$. This formula further generalizes to an arbitrary graph.

Upper bound f_c^{st} for the depinning threshold from the number of stable equilibria

We show that the above analysis can be extended for the computation of the number of *stable* equilibria in the presence of the external force. In order to consider only stable equilibria, the formula of the main text must be modified as follows

$$\mathcal{N}_{\text{st}} = \int_{\mathbb{R}^K} d\mathbf{u} \prod_{i=1}^K \left(e^{-u_i^2/(2w^2)} \delta(\partial_i \mathcal{H}[\mathbf{u}] - f_i) \right) \times \det(\partial_i \partial_j \mathcal{H}[\mathbf{u}]) \Theta(\partial_i \partial_j \mathcal{H}[\mathbf{u}]) \quad (75)$$

where $\Theta(M) = 1$ if all eigenvalues of the matrix M are positive, and 0 otherwise. Because $V'_i(u_i)$ and $V''_i(u_i)$ are independent (if the function $R(u)$ is symmetric), we have the same simplification as for the calculation of the mean total number of equilibria.

$$\langle \mathcal{N}_{\text{st}} \rangle = \langle \det(\partial_i \partial_j \mathcal{H}) \Theta(\partial_i \partial_j \mathcal{H}) \rangle \int_{\mathbb{R}^K} d\mathbf{u} \left\langle \prod_{i=1}^K \left(e^{-u_i^2/(2w^2)} \delta(\partial_i \mathcal{H}[\mathbf{u}] - f_i) \right) \right\rangle \quad (76)$$

where we have also made use of the fact that $\partial_i \partial_j \mathcal{H}$ does not depend explicitly on \mathbf{u} , but only through $V_i''(u_i)$, which are i.i.d. Gaussian random variables. Finally we can perform the same sequence of manipulations for a constant force

$$\langle \mathcal{N}_{\text{st}} \rangle = \underbrace{\langle \det(\partial_i \partial_j \mathcal{H}) \Theta(\partial_i \partial_j \mathcal{H}) \rangle}_{\sim \exp[L r_{\text{st}}]} \frac{\exp \left[-\frac{1}{2(w\kappa)^2} \mathbf{f}^T [(L_m^{-2} \mathbf{1}_K - \Delta)^2 + L_w^{-4} \mathbf{1}_K]^{-1} \mathbf{f} \right]}{\underbrace{|\det((L_m^{-2} + i L_w^{-2}) \mathbf{1}_K - \Delta)|}_{\sim \exp \left\{ -L \left[\text{re} \sqrt{\frac{1}{L_m^2} + \frac{1}{L_w^2}} + \frac{f^2}{2(m^4 w^2 + |R''(0)|)} \right] \right\}}} \quad (77)$$

where we have assumed the exponential behaviour in the absence of the external force $\langle \mathcal{N}_{\text{st}} \rangle|_{w=\infty} \sim e^{L r_{\text{st}}}$. This allows us to define a new rate controlling the number of stable equilibria $\langle \mathcal{N}_{\text{st}} \rangle \sim e^{L r_{\text{st}}(f)}$ and take the limits $\lim_{w \rightarrow \infty} \lim_{m^2 \rightarrow 0}$. Finally we obtain

$$r_{\text{st}}(f) = r_{\text{st}} - \frac{f^2}{2|R''(0)|}. \quad (78)$$

Despite the fact that we are not able to compute the rate r_{st} , we have identified its f dependence, which is simply quadratic as for the total number of equilibria. This result provides another, more stringent, upper bound for the depinning threshold as the value f_c^{st} of the force such that $r_{\text{st}}(f) = 0$ which gives $f_c^{\text{st}} = \sqrt{2r_{\text{st}}|R''(0)|} \leq f_c^{\text{tot}}$ as we have obviously $\mathcal{N}_{\text{st}} \leq \mathcal{N}_{\text{tot}}$.

C. PERTURBATIVE ANALYSIS OF THE GENERALISED LYAPUNOV EXPONENT

We present here the details on the derivation of the expansion (19) of the letter. We follow an idea introduced in [RT14] (section 6 of this reference) in a different situation: in the limit $E \rightarrow -\infty$ the process $z(x)$ is most of the time trapped near $z = \sqrt{-E}$, what suggests a perturbative expansion around the Ornstein-Uhlenbeck process. This can be most conveniently achieved at the level of the stochastic differential equation (SDE) (18) (this is more straightforward than from the Fokker-Planck equation (FPE)). For convenience, we write $E = -k^2$ and rescale the coordinate and the process as

$$z(x) = k(1 + \epsilon \zeta(u)) \quad \text{with} \quad u = kx. \quad (79)$$

Eq. (18) leads to the SDE in terms of dimensionless variables

$$\zeta'(u) = -2\zeta(u) + \eta(u) - \epsilon \zeta(u)^2, \quad (80)$$

where $\eta(u)$ is a normalised Gaussian white noise and

$$\epsilon = \sqrt{\frac{2D}{k^3}} \quad (81)$$

is the small perturbative parameter. We now expand the process in powers of ϵ : $\zeta = \zeta_0 + \zeta_1 + \zeta_2 + \dots$. The

different terms can be found recursively

$$\zeta_0(u) = \int_0^u dt e^{-2(u-t)} \eta(t) \quad (82)$$

$$\zeta_1(u) = -\epsilon \int_0^u dt e^{-2(u-t)} \zeta_0(t)^2 \quad (83)$$

$$\zeta_2(u) = 2\epsilon^2 \int_0^u dt e^{-2(u-t)} \zeta_0(t) \int_0^t dt' e^{-2(t-t')} \zeta_0(t')^2 \quad (84)$$

$$\zeta_3(u) = -\epsilon \int_0^u dt e^{-2(u-t)} [\zeta_1(t)^2 + 2\zeta_0(t)\zeta_2(t)] \quad (85)$$

etc. Making use of the Gaussian nature of the Ornstein-Uhlenbeck process and of

$$\langle \zeta_0(u) \zeta_0(u') \rangle_{\text{stat}} = \frac{1}{4} e^{-2|u-u'|} \quad (86)$$

we can compute any correlations functions. $\langle \dots \rangle_{\text{stat}}$ denotes averaging in the stationary regime (i.e. for $u, u' \gg 1$).

We can check the method on the Lyapunov exponent: it leads to the expansion

$$\frac{\gamma_1^{(\text{pert})}}{k} = 1 + \epsilon \langle \zeta(u) \rangle_{\text{stat}} = 1 - \frac{\epsilon^2}{8} - \frac{5\epsilon^4}{128} + \mathcal{O}(\epsilon^6) \quad (87)$$

which is in perfect correspondence with the analytic part of the expansion obtained from the exact result. The complex Lyapunov exponent for $D = 1$ is [LGP88]

$$\Omega = \gamma_1 - i\pi N = \frac{\text{Ai}'(-E) - i \text{Bi}'(-E)}{\text{Ai}(-E) - i \text{Bi}(-E)} \quad (88)$$

leading to

$$\frac{\gamma_1}{k} = 1 - \frac{\epsilon^2}{8} - \frac{5\epsilon^4}{128} + \mathcal{O}(\epsilon^6) - \frac{e^{-16/(3\epsilon^2)}}{2} \left[1 - \frac{\epsilon^2}{12} + \mathcal{O}(\epsilon^4) \right] \quad (89)$$

The Lyapunov exponent exhibits non-analytic contributions, $\sim \exp[-8|E|^{3/2}/(3D)]$, which were associated in the letter to the possibility of rare excursions to $\pm\infty$ (this problem was not present in the case studied in Ref. [RT14]).

The variance is given by

$$\gamma_2^{(\text{pert})} = 2k\epsilon^2 \lim_{u \rightarrow \infty} \langle \zeta(u) \int_0^u dv \zeta(v) \rangle_c \quad (90)$$

where $\langle XY \rangle_c = \langle XY \rangle - \langle X \rangle \langle Y \rangle$. The limit $u \rightarrow \infty$ ensures that the correlator is computed in the stationary regime. Some lengthy algebra gives

$$\frac{\gamma_2^{(\text{pert})}}{k} = \frac{\epsilon^2}{4} + \frac{9\epsilon^4}{64} + \mathcal{O}(\epsilon^6) \quad (91)$$

The leading term to the third cumulant is more easy to compute

$$\begin{aligned} \gamma_3^{(\text{pert})} &= k \epsilon^3 \lim_{u \rightarrow \infty} \frac{1}{u} \left\langle \left(\int^u (\zeta - \langle \zeta \rangle) \right)^3 \right\rangle \\ &= 3k \epsilon^3 \lim_{u \rightarrow \infty} \frac{1}{u} \left\langle \left(\int^u \zeta_0 \right)^2 \int^u (\zeta_1 - \langle \zeta_1 \rangle) \right\rangle + \dots \end{aligned} \quad (92)$$

hence

$$\frac{\gamma_3^{(\text{pert})}}{k} = -\frac{3\epsilon^4}{16} + \mathcal{O}(\epsilon^6) \quad (93)$$

(it is easy to see that $\gamma_4 = \mathcal{O}(\epsilon^6)$).

Gathering (89,91,93) leads to (19).

D. NUMERICAL CALCULATIONS

Details on the numerical results presented in the letter are given here. We propose two different and complementary numerical methods allowing to determine the generalised Lyapunov exponent (GLE) $\Lambda(q)$. Both are based on the spectral analysis of the non-Hermitian operator $\mathcal{O}_q = \mathcal{G}^\dagger + qz$ controlling the evolution of the moments $\langle |y(L)|^q \rangle$. The spectral analysis leads to introduce the biorthogonal set of right and left eigenvectors

$$(\mathcal{G}^\dagger + qz) \Phi_n^R(z; q) = -\mathcal{E}_n(q) \Phi_n^R(z; q) \quad (94)$$

$$(\mathcal{G} + qz) \Phi_n^L(z; q) = -\mathcal{E}_n(q) \Phi_n^L(z; q) \quad (95)$$

where $n = 0, 1, 2, \dots$, with normalisation condition $\int dz \Phi_n^L(z; q) \Phi_m^R(z; q) = \delta_{n,m}$. Here we write the forward generator as

$$\mathcal{G}^\dagger = D \frac{d^2}{dz^2} + \frac{d}{dz} \mathcal{U}'(z) = D \frac{d}{dz} e^{-\mathcal{U}(z)/D} \frac{d}{dz} e^{\mathcal{U}(z)/D} \quad (96)$$

where $\mathcal{U}(z) = Ez + z^3/3$. The GLE thus coincides with the lowest eigenvalue $\Lambda(q) = -\mathcal{E}_0(q)$. We first discretize the operator and perform a direct diagonalisation. The second method is based on a neat analysis of the differential equation.

D.1. Continuous time random walk with drift

The FPE $\partial_\tau \mathcal{P}_\tau(z) = \mathcal{G}^\dagger \mathcal{P}_\tau(z)$ can be discretized in space as follows : we write $z = nb$ and introduce the transition rate from site m to site $n = m \pm 1$

$$t_{n,m} = \frac{1}{b^2} e^{(\mathcal{U}_m - \mathcal{U}_n)/2} \quad (97)$$

where $\mathcal{U}_n = \mathcal{U}(nb)$. The continuous time random walk is thus described by

$$\begin{aligned} \partial_\tau P_\tau(n) &= t_{n,n+1} P_\tau(n+1) + t_{n,n-1} P_\tau(n-1) \\ &\quad - (t_{n+1,n} + t_{n-1,n}) P_\tau(n) \end{aligned} \quad (98)$$

(in the limit $b \rightarrow 0$ we recover the continuous diffusion for $D = 1$; cf. Ref. [HT08] for example). The boundaries must be discussed in detail. In order to mimic the absorption at $z = -\infty$ with reinjection at $z = +\infty$, we consider a finite lattice $n \in \{-L, -L+1, \dots, +L\}$ and choose the following transition rates connecting the two boundaries

$$t_{L,-L} = \frac{1}{b^2} p_{\text{inj}} \quad \text{and} \quad t_{-L,L} = 0 \quad (99)$$

where p_{inj} is a parameter. Eq. (98) must be supplemented by

$$\begin{aligned} \partial_\tau P_\tau(-L) &= t_{-L,-L+1} P_\tau(-L+1) \\ &\quad - (t_{-L+1,-L} + t_{L,-L}) P_\tau(-L) \end{aligned} \quad (100)$$

$$\begin{aligned} \partial_\tau P_\tau(L) &= t_{L,L-1} P_\tau(L-1) + t_{L,-L} P_\tau(-L) \\ &\quad - t_{L-1,L} P_\tau(L). \end{aligned} \quad (101)$$

These equations define the $(2L+1) \times (2L+1)$ matrix $(\mathcal{G}^\dagger)_{n,m}$. Adding $(qn)b\delta_{n,m}$ we obviously obtain the matrix $(\mathcal{O}_q)_{n,m}$. We perform exact diagonalization.

We first check the method in the small D/k^3 (perturbative) limit for $q \neq 1$: Fig. 2 shows that the agreement with the perturbative result is excellent already for $D/k^3 = 2/3$ (for $b = 0.005$ and $L = 1000$). The study of the case $q = 1$ is more tricky as it requires to identify an exponentially small non-analytic correction. First we have checked that the finite size effect is negligible (provided $L \times b \gg k$), as well as the role of the parameter p_{inj} (it can be changed by several orders of magnitude without modifying significantly the result). The most important parameter is the lattice spacing b . Fig. 4 shows the different data obtained by diminishing b while keeping $L \times b$ constant. This confirms : (i) that $r = \Lambda(1) - k$ is non analytic in the small parameter D/k^3 . (ii) its main exponential behaviour is $\sim \exp[-4k^3/(3D)]$.

D.2. Analysis of the differential equation

$$\mathcal{O}_q \Phi_0^R = \Lambda(q) \Phi_0^R$$

The numerical study of the rate (17) at large k requires to determine an eigenvalue with extreme precision. For Hermitian one-dimensional operators, such as Schrödinger operators, a well-known algorithm, based on the Strum-Liouville theorem stating that the n -th excited state has n nodes, provides the spectrum with high accuracy. At first sight this method seems applicable as it is possible to map the non-Hermitian operator \mathcal{O}_q onto an

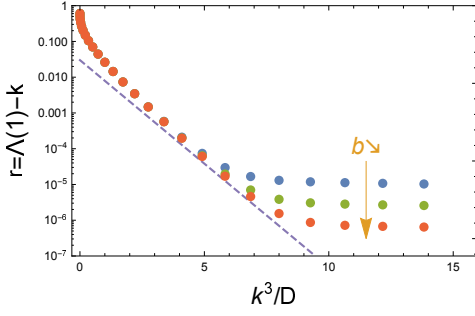


FIG. 4: Rate r obtained by diagonalization of $(2L + 1) \times (2L + 1)$ matrix $(\mathcal{O}_q)_{n,m}$. Lattice spacing is $b = 0.02, 0.01$ and 0.005 , with $L \times b = 5$ fixed ($0 \leq k \leq 2.4$). The line corresponds to $\propto \exp[-4|E|^{3/2}/(3D)]$.

effective Schrödinger Hermitian operator thanks to the non-unitary transformation

$$\mathcal{H}_q = -e^{\mathcal{U}(z)/(2D)} \mathcal{O}_q e^{-\mathcal{U}(z)/(2D)} \quad (102)$$

leading to

$$\mathcal{H}_q = -D e^{\mathcal{U}(z)/(2D)} \frac{d}{dz} e^{-\mathcal{U}(z)/D} \frac{d}{dz} e^{\mathcal{U}(z)/(2D)} - qz \quad (103)$$

$$= -D \frac{d^2}{dz^2} + \frac{[\mathcal{U}'(z)]^2}{4D} - \frac{\mathcal{U}''(z)}{2} - qz \quad (104)$$

(such a transformation is well-known in the context of the FPE, see for instance [R89]). The right eigenvector of \mathcal{O}_q is thus related to an eigenfunction of \mathcal{H}_q by $\Psi_n(z; q) = \Phi_n^R(z; q) e^{\mathcal{U}(z)/(2D)}$. In the present case, we are however faced with a difficulty as the *spectra of the two operators \mathcal{H}_q and \mathcal{O}_q differ*. This can be demonstrated easily for $q = 0$: the first eigenvector of the forward generator satisfies $\mathcal{G}^\dagger \Phi_0^R(z; 0) = 0$. This is the stationary state $\Phi_0^R(z; 0) = (N/D) e^{-\mathcal{U}(z)/D} \int_{-\infty}^z dz' e^{\mathcal{U}(z')/D}$, where $N(E)$ is the integrated DoS, measuring also the probability current (we have also $\Phi_0^L(z; 0) = 1$). The inspection of its asymptotic behaviour

$$\Phi_0^R(z; 0) \simeq N z^{-2} \quad \text{for } z \rightarrow \pm\infty, \quad (105)$$

which illustrates that the drift $\mathcal{U}'(z) \simeq -z^2$ is dominant at large z , makes clear that $\Psi_0(z; 0) = \Phi_0^R(z; 0) e^{\mathcal{U}(z)/(2D)}$ is not normalisable, thus $\mathcal{E}_0(0) = 0$ does not belong to the spectrum of \mathcal{H}_0 . This observation has its origin in the non confining character of the potential $\mathcal{U}(z)$: the stationary state describes a *non-equilibrium* situation with a finite current (in the case of a confining potential leading to equilibrium, the two operators share the same spectrum). Correspondingly, the supersymmetric operator $\mathcal{H}_0 = \mathcal{Q}^\dagger \mathcal{Q}$ where

$$\mathcal{Q} = -\sqrt{D} e^{-\mathcal{U}(z)/(2D)} \frac{d}{dz} e^{\mathcal{U}(z)/(2D)}, \quad (106)$$

has a strictly positive ground state $\mathcal{E}_1(0) > 0$ and one says that the supersymmetry is *broken*. This observation also holds for $q > 0$, as we have checked numerically by direct diagonalization of the two operators:

$$\text{Spec}(-\mathcal{O}_q) = \text{Spec}(\mathcal{H}_q) \cup \{\mathcal{E}_0(q)\}. \quad (107)$$

A practical consequence of this observation is that both $\Phi_0^R(z; q)$ and $\Phi_1^R(z; q)$ are strictly positive, which makes difficult to identify $\mathcal{E}_0(q)$: in particular one can not use the node method as the first node only appears for energy above $\mathcal{E}_1(q)$.

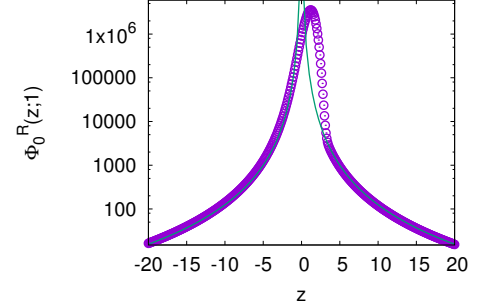


FIG. 5: Function $\Phi_0^R(z; q)$ (not normalised) for $k = 1$ and $q = 1$. The line is $\propto |x|^{-3}$.

a. Ground state $\mathcal{E}_0(q)$ of $-\mathcal{O}_q$

We have found a criterion allowing to determine $\Lambda(q) = -\mathcal{E}_0(q)$ by a careful study of the function $\Phi_0^R(z; q)$ obtained numerically (Fig. 5). The starting point is the differential equation

$$\varphi''(z) + (E + z^2)\varphi'(z) + [(2 + q)z - \lambda]\varphi(z) = 0 \quad (108)$$

corresponding to $\mathcal{O}_q \Phi_0^R = \Lambda(q) \Phi_0^R$ for arbitrary value of the spectral parameter (and for $D = 1$). A spectral problem is defined by specifying boundary conditions (or asymptotic behaviours). Here, it is easy to see that the two linearly independent asymptotic behaviours are $|z|^{-2-q}$ and $|z|^q \exp[-\mathcal{U}(z)]$. For all eigenvalues of \mathcal{H}_q , i.e. $\lambda = -\mathcal{E}_n(q)$ for $n > 0$, the eigenfunctions $\Psi_n(z; q)$ decay exponentially, which is related to the behaviours $\varphi(z) = \Phi_n^R(z; q) \sim |z|^q \exp[-\mathcal{U}(z)]$ for $z \rightarrow +\infty$ and $|z|^{-2-q}$ for $z \rightarrow -\infty$. Only the first right eigenvector presents an algebraic behaviour on both sides $\Phi_0^R(z; q) \sim |z|^{-2-q}$, which defines the spectral problem (108) leading to $\lambda = \Lambda(q)$. The crucial observation, which we have made numerically, is that the coefficients controlling the two tails are equal, as in the case $q = 0$, see Eq. (105) (Fig. 5). Based on this observation, we propose the following method: we solve (108) and choose λ large enough to get the power law behaviours

$$\varphi(z) \simeq A_{\pm}(\lambda) |z|^{-2-q} \quad \text{for } z \rightarrow \pm\infty \quad (109)$$

The value of the eigenvalue is given by finding where the two coefficients match exactly

$$A_-(\lambda) = A_+(\lambda) \quad \text{for} \quad \lambda = \Lambda(q). \quad (110)$$

We have compared the numerical values obtained in this way with the one deduced by direct diagonalization of the matrix $(\mathcal{O}_q)_{n,m}$ (previous subsection): we have obtained a perfect agreement for smaller values of the parameter k , when diagonalization is reliable (Fig. 3). Moreover the method is sufficiently accurate to make accessible relatively large k (up to $k = 2.5$) leading to rate of order 10^{-11} . In particular it allows to extract unambiguously the behavior of the pre-exponential function of the rate: in Fig. 6 we fit $r \times \exp[4|E|^{3/2}/(3D)]$ by $\simeq 0.08(D/|E|)$.

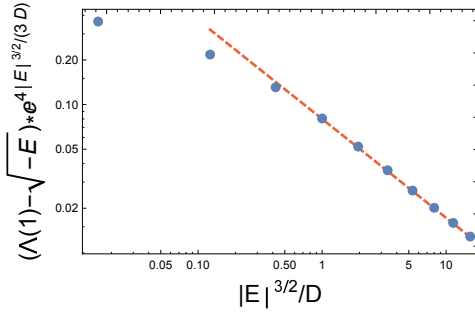


FIG. 6: Pre-exponential function of the rate r , extracted from the same numerical data as in Fig. 3 in the text. The dashed line corresponds to $0.08D/|E|$.

b. Ground state $\mathcal{E}_1(q)$ of \mathcal{H}_q

The ground state energy of the Hamiltonian \mathcal{H}_q is also of interest as it controls the relaxation towards equilibrium. We have obtained it by a direct diagonalization of the discretized operator \mathcal{H}_q : (we also checked that the diagonalization of $(\mathcal{O}_q)_{n,m}$ gives the same result): we have plotted the three first energies in Fig. 7. A more precise method is to study neatly the differential equation (108): the first node in the solution $\varphi(z)$ appears when $\lambda \leq -\mathcal{E}_1(q)$. This allows to determine the non-analytic contribution to $\mathcal{E}_1(q)$ for relatively large $|E|^{3/2}/D$.

The two energy levels become extremely close in the large $k/D^{1/3}$ limit (cf. Fig. 7). We gave above $-\mathcal{E}_0(1) \simeq k + 0.08(D/|E|) \exp[-4|E|^{3/2}/(3D)]$ (Fig. 6). A fine analysis of $\mathcal{E}_1(1)$ shows that the non-analytic contribution to $\mathcal{E}_1(1)$ is *half* of the correction to $\mathcal{E}_0(1)$: $-\mathcal{E}_1(1) \simeq k + 0.04(D/|E|) \exp[-4|E|^{3/2}/(3D)]$ (Fig. 8). Below it will be shown that such non-analytic contribution to $\mathcal{E}_1(1)$ can indeed be obtained by a judicious extension of the classical WKB analysis.

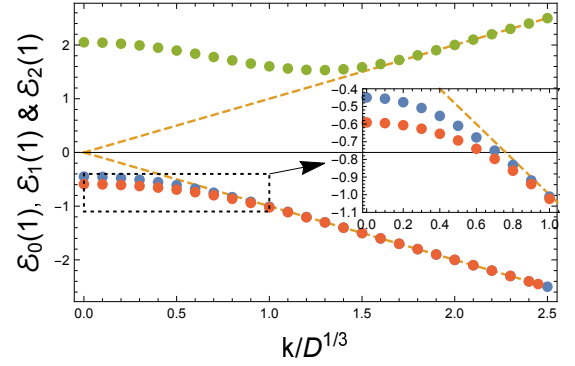


FIG. 7: The three first energies obtained by diagonalization for $q = 1$. Straight dashed lines are $\pm k$

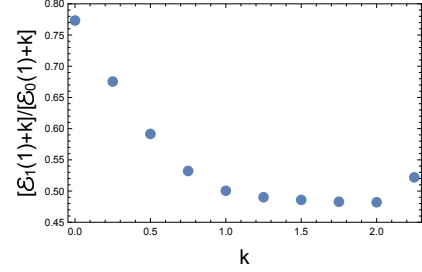


FIG. 8: Non analytic contributions to the two first energies obtained by analysis of the differential equation (108).

E. WKB ANALYSIS OF $\mathcal{E}_1(q)$

The generalised Lyapunov exponent $\Lambda(q) = -\mathcal{E}_0(q)$ studied in the letter is related to the largest eigenvalue of the operator $\mathcal{O}_q = \mathcal{G}^\dagger + qz$. In this section we study the next eigenvalue, $\mathcal{E}_1(q)$, which controls finite size effect (properties of the elastic line of finite length L), and also provides a lower bound for the rate of interest in the letter. The calculation confirms the behaviour obtained numerically in the previous section for the $D \rightarrow 0$ behaviour.

E.1. The double well effective potential

We analyse the ground state energy $\mathcal{E}_1(q)$ of the Hamiltonian \mathcal{H}_q introduced in Section D. We find convenient to rescale the coordinate as $\zeta = |E|^{-1/2}z$ and the energy as $\mathcal{H}_q = |E|^{-1/2}\mathcal{H}_q$, hence we denote $\mathcal{E}_n(q) = |E|^{-1/2}\mathcal{E}_n(q) = L_m\mathcal{E}_n(q)$ ($n \in \mathbb{N}^*$) the spectrum of the new Hamiltonian

$$\mathcal{H}_q = -s \frac{d^2}{d\zeta^2} + V_q(\zeta) \quad (111)$$

where

$$V_q(\zeta) = \frac{1}{4s}(\zeta^2 - 1)^2 - (q+1)\zeta \quad (112)$$

(see Fig. 9). One can show that the eigenvalues of \mathcal{H}_q are $\mathcal{E}_n(q)$ with $n = 1, 2, \dots$. Note that the spectra of $-\mathcal{O}_q$ and \mathcal{H}_q coincide apart from the eigenvalue $\mathcal{E}_0(q)$, which is in the spectrum of \mathcal{O}_q but not of \mathcal{H}_q , as discussed in the previous Section. Note that for $q = 0$, the Hamiltonian (111) has the correct expected exact non-normalizable eigenstate at zero energy.

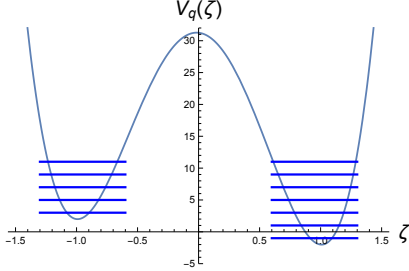


FIG. 9: The effective potential for $q = 1$. The two spectra associated with the two harmonic wells are in correspondence for integer q (apart for the $q + 1$ lowest levels).

As we have seen in the letter, the results seem to give a strong evidence in favour of very fast, hence non-perturbative vanishing of the $\Lambda(1) = -|E|^{1/2}\mathcal{E}_0(1)$ in this regime. Although $\Lambda(1)$ is given by an eigenvalue which does not belong to the spectrum of \mathcal{H}_q , we still can analyze semiclassically the ground state energy $\mathcal{E}_1(q)$ of the latter Hamiltonian, which provides a lower bound for $\Lambda(q) = -\mathcal{E}_0(q) > -\mathcal{E}_1(q)$ (moreover the numerics has showed that the non-analytic corrections to $\mathcal{E}_0(1)$ and $\mathcal{E}_1(1)$ only differ by a factor 1/2 in the $s \rightarrow 0$ limit ; cf. Section D). The only way the energy $\mathcal{E}_1(q)$ of the ground state (located semiclassically in the right well around $\zeta \approx 1$) can get such a non-perturbative shift is by a tunneling admixture from the lowest-level eigenstate located semiclassically in the higher (left) potential well.

We now develop an improved WKB procedure in order to analyse the ground state of the Schrödinger equation

$$\mathcal{H}_q \Psi(\zeta) = \mathcal{E} \Psi(\zeta) \quad (113)$$

as $s \rightarrow 0$. The potential $V_q(\zeta)$ has two deep minima around $\zeta = \pm 1$ separated by a big barrier around $\zeta = 0$ (Fig. 9). The potential is well approximated by two quadratic wells near these two points

$$V_q(\zeta) \simeq q + 1 + \frac{1}{s}(\zeta + 1)^2 \quad \text{for } \zeta \sim -1 \quad (114)$$

$$\simeq -(q + 1) + \frac{1}{s}(\zeta - 1)^2 \quad \text{for } \zeta \sim +1. \quad (115)$$

Before starting the presentation of the WKB method, it is useful to have in mind the correspondence with the standard formulae in quantum mechanics. This is done by identifying $s = \hbar^2/2m$: we can set $\hbar = 1$ and $m = 1/(2s)$. We see that the classical oscillator frequency in each well in our problem is given by $\omega = 2$. The spectra

related to the two harmonic wells (114,115) are :

$$\begin{cases} \mathcal{E}_n^R = 2n - q \\ \mathcal{E}_n^L = 2n + 2 + q \end{cases} \quad \text{with } n \in \mathbb{N} \quad (116)$$

which shows that the two spectra are in correspondence for integer q , apart for the $q + 1$ lowest levels (Fig. 9).

E.2. Weakly asymmetric double well ($|q + 1| \ll 1$)

For $q = -1$ the double well potential is symmetric and it is possible to use known results to express the energy of the ground state [LL66, G00, S08]. The analysis of the bottom of the spectrum can be mapped onto a simple two level problem

$$\begin{pmatrix} \mathcal{E}_0^R & -\Delta/2 \\ -\Delta/2 & \mathcal{E}_0^L \end{pmatrix} \quad (117)$$

For $q = -1$, we have $\mathcal{E}_0^R = \mathcal{E}_0^L$ and the two eigenvalues are $\pm\Delta/2$. Going back to standard notations, the semiclassical calculation of the splitting for the Hamiltonian $H = -[\hbar^2/(2m)]\partial_x^2 + V(x)$, with $V(x) = (1/8)ma^2\omega^2[(x/a)^2 - 1]^2$, is [G00] $\Delta = 4\sqrt{3}\hbar\omega\sqrt{S/(2\pi\hbar)}\exp[-S/\hbar]$ where $S = \int_{-a}^a dx \sqrt{2mV(x)}$ (the formula given in § 50 of [LL66] must be corrected by a factor $\sqrt{\pi/e}$ [G00]), which gives the ground state energy

$$\mathcal{E}_1(-1) \simeq 1 - \frac{\Delta}{2} \simeq 1 - \frac{4}{\sqrt{\pi s}} e^{-2/(3s)}. \quad (118)$$

If q is still close to -1 but slightly deviates such that $|\mathcal{E}_0^L - \mathcal{E}_0^R| = 2|q + 1| \gg \Delta \sim \exp[-2/(3s)]$, we can assume that the coupling is still described by the same formula and use second order perturbation formula [T15]. As a consequence the shift of energy levels is now much smaller : whereas it was proportional to the tunneling amplitude $\sim \exp[-S/\hbar]$ in the degenerate case, it is now proportional to the tunneling probability $\sim \exp[-2S/\hbar] \sim \exp[-4/(3s)]$:

$$\mathcal{E}_1(q) \simeq \mathcal{E}_0^R - \frac{|\Delta/2|^2}{\mathcal{E}_0^L - \mathcal{E}_0^R} \underset{q \simeq -1}{\simeq} -q - \frac{2}{\pi(q + 1)s} e^{-4/(3s)}. \quad (119)$$

We must however remember that (118) and (119) only provides the non-analytic contributions (in s) to the ground state energy, which also receives (analytic) contributions due to the anharmonicity of the potential (of the same origin as the contribution $-\Lambda_{\text{pert}}(q)$ to $\mathcal{E}_0(q)$).

E.3 Strongly asymmetric double well ($|q + 1| \gtrsim 1$)

In the “strongly” asymmetric potential limit, i.e. for $|q + 1| \gtrsim 1$, it is not anymore possible to restrict the

problem to a two level problem. We have to develop a different strategy involving two different approximation schemes : in the neighbourhood of each potential well ($\zeta \sim \pm 1$), the potential is replaced by parabolas, which allows to express the exact solution of the approximated Schrödinger equation locally. In between, inside the potential barrier, we write the approximate WKB solution of the (exact) Schrödinger equation and match the three expressions. This strategy is borrowed from Ref. [S08].

Step 1: wave function for $\zeta \sim +1$

In the neighbourhood of the right well, the Schrödinger equation (113) takes the form

$$\left[-s \frac{d^2}{d\zeta^2} - q - 1 + \frac{1}{s}(\zeta - 1)^2 \right] \Psi(\zeta) \simeq \mathcal{E} \Psi(\zeta). \quad (120)$$

It will be convenient to introduce the variable $\xi = (\zeta - 1)/\sqrt{s}$ and parametrize the energy as $\mathcal{E} = \mathcal{E}_0^R + 2\epsilon = -q + 2\epsilon$, where ϵ is a small (negative) shift to the ground state energy of the right well. Extracting the Gaussian function $\Psi(\zeta) = y(\xi) \exp[-(1/2)\xi^2]$ we obtain that the function $y(\xi)$ obeys the Hermite equation $y''(\xi) - 2\xi y'(\xi) + 2\epsilon y(\xi) = 0$, whose solution can be expressed in terms of the Hermite function [NO83] with integral representation

$$H_\epsilon(\xi) = \frac{1}{\Gamma(-\epsilon)} \int_0^\infty dt t^{-1-\epsilon} e^{-t^2-2\xi t} \quad (121)$$

(for $\epsilon < 0$). This integral representation is suitable to extract the asymptotic behaviour

$$H_\epsilon(\xi) \simeq (2\xi)^\epsilon \quad \text{for } \xi \rightarrow +\infty \quad (122)$$

and, splitting the integral in (121) as $\int_0^\infty = \int_{-\infty}^{+\infty} - \int_{-\infty}^0$:

$$H_\epsilon(\xi) \simeq -(-2\xi)^\epsilon + \frac{\sqrt{\pi}}{\Gamma(-\epsilon)} (-\xi)^{-\epsilon-1} e^{+\xi^2} \quad \text{for } \xi \rightarrow -\infty. \quad (123)$$

This shows that the solution of (120),

$$\Psi(\zeta) \simeq C_R H_\epsilon \left(\frac{\zeta - 1}{\sqrt{s}} \right) e^{-(\zeta-1)^2/(2s)}, \quad (124)$$

decays exponentially for $\xi = (\zeta - 1)/\sqrt{s} \rightarrow +\infty$ and grows exponentially for $\xi = (\zeta - 1)/\sqrt{s} \rightarrow -\infty$. Keeping the variable ξ , we write :

$$\begin{aligned} \Psi(\zeta) &\simeq C_R \\ &\times \begin{cases} -(-2\xi)^\epsilon e^{-\xi^2/2} + \frac{\sqrt{\pi}}{\Gamma(-\epsilon)} (-\xi)^{-\epsilon-1} e^{+\xi^2/2} & \text{as } \xi \rightarrow -\infty \\ (2\xi)^\epsilon e^{-\xi^2/2} & \text{as } \xi \rightarrow +\infty \end{cases} \end{aligned} \quad (125)$$

Step 2: wave function for $\zeta \sim -1$

In the neighbourhood of the left harmonic well, the Schrödinger equation (113) takes the form

$$\left[-s \frac{d^2}{d\zeta^2} + q + 1 + \frac{1}{s}(\zeta + 1)^2 \right] \Psi(\zeta) \simeq \mathcal{E} \Psi(\zeta). \quad (126)$$

It is now convenient to write the energy as $\mathcal{E} = \mathcal{E}_0^L + 2\tilde{\epsilon} = q + 2 + 2\tilde{\epsilon}$ so that $\tilde{\epsilon} = -(q+1) + \epsilon \simeq -(q+1)$, as we expect that ϵ is exponentially small. We choose the solution

$$\Psi(\zeta) \simeq C_L H_{\tilde{\epsilon}} \left(-\frac{\zeta + 1}{\sqrt{s}} \right) e^{-(\zeta+1)^2/(2s)}, \quad (127)$$

which now decays in the negative direction and grows in the positive direction :

$$\begin{aligned} \Psi(\zeta) &\simeq C_L \\ &\times \begin{cases} (-2\xi)^{\tilde{\epsilon}} e^{-\xi^2/2} & \text{as } \xi \rightarrow -\infty \\ -(-2\xi)^{\tilde{\epsilon}} e^{-\xi^2/2} + \frac{\sqrt{\pi}}{\Gamma(-\tilde{\epsilon})} \xi^{-\tilde{\epsilon}-1} e^{+\xi^2/2} & \text{as } \xi \rightarrow +\infty \end{cases} \end{aligned} \quad (128)$$

where now $\xi = (\zeta + 1)/\sqrt{s}$.

Step 3: WKB solution inside the barrier

Inside the potential barrier, the solution of (113) is well approximated by the WKB wave function

$$\begin{aligned} \Psi(\zeta) &\simeq \Psi^{\text{WKB}}(\zeta) = \left(\frac{\sqrt{s(2q+1)}}{2s p(\zeta)} \right)^{1/2} \\ &\times \left[A e^{\int_{\zeta_1}^{\zeta} d\zeta' p(\zeta')} + B e^{-\int_{\zeta_1}^{\zeta} d\zeta' p(\zeta')} \right] \end{aligned} \quad (129)$$

where

$$p(\zeta) = \sqrt{\frac{1}{s}(V_q(\zeta) - \mathcal{E})}. \quad (130)$$

The validity of the WKB expression (129) extends to the domain at the left of the turning point ζ_0 where $\mathcal{E} = V_q(\zeta_0)$. Using the approximate form (115) we find $\zeta_0 = 1 - \sqrt{s}[1 + \epsilon + \mathcal{O}(\epsilon^2)]$. Anticipating, since we expect that ϵ is exponentially small, $\sim e^{-4/(3s)}$, we have

$$\zeta_0 = 1 - \sqrt{s} + \mathcal{O}(e^{-4/(3s)}). \quad (131)$$

Step 4: matching

In order to match the WKB solution (129) with (124) and (127), it is convenient to introduce a specific notation for the action : we define

$$\Phi_L(\zeta) = \int_{-1}^{\zeta} p(\zeta') d\zeta' \quad \text{and} \quad \Phi_R(\zeta) = \int_{\zeta}^{\zeta_0} p(\zeta') d\zeta'. \quad (132)$$

The total action associated to the trajectory going from $\zeta = -1$ to the turning point is denoted

$$S_q = \int_{-1}^{\zeta_0} p(\zeta') d\zeta' \quad (133)$$

thus $S_q = \Phi_L(\zeta) + \Phi_R(\zeta)$, obviously.

Matching of (124) with (129).— In the vicinity of the right harmonic well at $\zeta = +1$, the asymptotic form (125) for $\xi \rightarrow -\infty$ should match with the WKB solution (129), which provides a first constraint on the two coefficients A and B . We must therefore consider ζ sufficiently far from the turning point, i.e. $(\zeta_0 - \zeta) \gg \sqrt{s}$, so that the asymptotic behaviour (125) holds. ζ must however be sufficiently close to the turning point so that the parabolic approximation (for the potential) is still justified. This last observation allows us to simplify the action $\Phi_R(\zeta)$: the parameter $t_* = (1 - \zeta)/(1 - \zeta_0) \gg 1$ will be treated as a large parameter. Using $V_q(\zeta_0) = \mathcal{E}$ we have $p(\zeta) = (1/s)\sqrt{(1 - \zeta)^2 - (1 - \zeta_0)^2}$. Introducing the variable $t = (1 - \zeta')/(1 - \zeta_0)$ we find in this approximation:

$$\begin{aligned} \Phi_R(\zeta) &= \frac{1}{s} \int_{\zeta}^{\zeta_0} \sqrt{(\zeta' - 1)^2 - (\zeta_0 - 1)^2} d\zeta' \quad (134) \\ &= \frac{(1 - \zeta_0)^2}{s} \int_1^{t_*} \sqrt{t^2 - 1} dt \\ &= \frac{(1 - \zeta_0)^2}{s} \frac{1}{2} \left(t_* \sqrt{t_*^2 - 1} - \operatorname{arccosh}(t_*) \right) \end{aligned}$$

and finally using $t_* \sqrt{t_*^2 - 1} \approx t_*^2 - 1/2$ and $\operatorname{arccosh}(t_*) \approx \ln(2t_*)$, we get

$$\Phi_R(\zeta) \simeq \frac{(1 - \zeta_0)^2}{2s} \left[t_*^2 - \ln(2t_*) - \frac{1}{2} \right].$$

We now find convenient to express $\Phi_R(\zeta)$ in terms of the variable $\xi = (\zeta - 1)/\sqrt{s} = -[(1 - \zeta_0)/\sqrt{s}] t_*$ introduced above :

$$\Phi_R(\zeta) \simeq \frac{1}{2} \xi^2 - \frac{1 + 2\epsilon}{2} \ln \left(\frac{-2\xi}{\sqrt{1 + 2\epsilon}} \right) - \frac{1 + 2\epsilon}{4} \quad (135)$$

from which we write the WKB wave function as

$$\begin{aligned} \Psi^{\text{WKB}}(\zeta) &\simeq A (2q + 1)^{1/4} e^{S_q + 1/4} (-2\xi)^\epsilon e^{-\xi^2/2} \\ &\quad + B (2q + 1)^{1/4} e^{-S_q - 1/4} (-2\xi)^{-1-\epsilon} e^{+\xi^2/2}, \end{aligned} \quad (136)$$

where we made use that $\epsilon \ll 1$. Matching with (125) for $\xi \rightarrow -\infty$ gives

$$A = -e^{-S_q - 1/4} C_R \quad (137)$$

$$B = -2\epsilon\sqrt{\pi} e^{S_q + 1/4} C_R \quad (138)$$

where we have used $\Gamma(-\epsilon) \simeq -1/\epsilon$. We get the first condition

$$\frac{A}{B} = \frac{e^{-1/2}}{2\epsilon\sqrt{\pi}} e^{-2S_q}. \quad (139)$$

Matching of (127) with (129).— We proceed the same way in the neighbourhood of the left harmonic well and match the asymptotic behaviour (128) for $\xi \rightarrow +\infty$ with the WKB approximation (129). In the region where the two expressions of the wave function match, we can use the parabolic approximation for the potential :

$$\Phi_L(\zeta) = \frac{1}{s} \int_{-1}^{\zeta} \sqrt{(2q + 1)s + (\zeta' + 1)^2} d\zeta' \quad (140)$$

(we recall that we can neglect ϵ there). Introducing the variable $t = (\zeta' + 1)/\sqrt{s(2q + 1)}$, this rewrites

$$\begin{aligned} \Phi_L(\zeta) &= (2q + 1) \int_0^{t_*} \sqrt{1 + t^2} dt \\ &= (2q + 1) \frac{1}{2} \left(t_* \sqrt{1 + t_*^2} + \operatorname{arcsinh}(t_*) \right) \end{aligned}$$

(129) and (128) match in the regime where $t_* = (\zeta + 1)/\sqrt{s(2q + 1)} \gg 1$, thus, using $t_* \sqrt{1 + t_*^2} \approx t_*^2 + 1/2$ and $\operatorname{arcsinh}(t_*) \approx \ln(2t_*)$, we find

$$\Phi_L(\zeta) \simeq \left(q + \frac{1}{2} \right) \left(t_*^2 + \ln(2t_*) + \frac{1}{2} \right).$$

At this stage it is convenient to use the variable $\xi = (\zeta + 1)/\sqrt{s} = \sqrt{(2q + 1)} t_*$ introduced above :

$$\begin{aligned} \Psi^{\text{WKB}}(\zeta) &\simeq A e^{(2q+1)/4} \left(\frac{2\xi}{\sqrt{(2q+1)}} \right)^q e^{+\xi^2/2} \\ &\quad + B e^{-(2q+1)/4} \left(\frac{2\xi}{\sqrt{(2q+1)}} \right)^{-q-1} e^{-\xi^2/2} \end{aligned} \quad (141)$$

which matches with (128) for $\xi \rightarrow +\infty$ (we recall that $\tilde{\epsilon} \simeq -q - 1$) if

$$A = \frac{2^{-q}\sqrt{\pi}}{\Gamma(q+1)} (2q+1)^{q/2} e^{-(2q+1)/4} C_L \quad (142)$$

$$B = -(2q+1)^{-(q+1)/2} e^{(2q+1)/4} C_L \quad (143)$$

We deduce the second condition

$$\frac{A}{B} = -\frac{2^{-q}\sqrt{\pi}}{\Gamma(q+1)} (2q+1)^{q+1/2} e^{-(2q+1)/2}. \quad (144)$$

It is worth emphasizing the result $A \sim B$, obtained from the improved WKB method developed here. The naive (standard) WKB method would have given $B|_{\text{naive WKB}} = 0$, since the only term of (129) which is vanishing as $\zeta \rightarrow -\infty$, in the classically forbidden region, is the term with coefficient A .

Conclusion : Ground state energy

Comparing (139) and (144) finally provides the expression of the shift of the ground state energy

$$\epsilon \simeq -\frac{\Gamma(q+1) e^q}{2\pi\sqrt{2}(q+1/2)^{q+1/2}} e^{-2S_q} \quad (145)$$

Making use of (131), we can now analyse the action S_q . For this purpose it is convenient to write $S_q = F_q(s)/(2s)$ with

$$F_q(s) = \int_{-1}^{1-\sqrt{s}} d\zeta \sqrt{(1-\zeta^2)^2 - 4s(q+1)\zeta + 4sq}. \quad (146)$$

Obviously $F_q(0) = 2/3$, which gives $S_q \simeq 2/(3s)$ as $s \rightarrow 0$. The correction to this main behaviour is obtained by considering $F'_q(s)$ which can be shown to behave logarithmically as $F'_q(s) \simeq q \ln s$. As a result

$$S_q \underset{s \rightarrow 0}{\simeq} \frac{2}{3s} - \frac{q}{2} \ln s + c_q, \quad (147)$$

where the constant c_q is studied numerically ($c_1 \simeq 1.05$). We conclude that the ground state energy of the Schrödinger operator \mathcal{H}_q is given by

$$\begin{aligned} \mathcal{E}_1(q) &= -q + 2\epsilon \\ &\simeq -q - \frac{\Gamma(q+1) e^{-2c_q+q}}{\pi\sqrt{2}(q+1/2)^{q+1/2}} s^q e^{-4/(3s)} \end{aligned} \quad (148)$$

which presents a different pre-exponential dependence, compared to (119) obtained for weakly asymmetric double-well. Going back to the initial notation

$$\mathcal{E}_1(q) \simeq -q\sqrt{|E|} - \alpha_q \frac{D^q}{|E|^{(3q-1)/2}} e^{-4|E|^{3/2}/(3D)}. \quad (149)$$

where

$$\alpha_q = \frac{\Gamma(q+1) e^{-2c_q+q}}{\pi\sqrt{2}(q+1/2)^{q+1/2}}. \quad (150)$$

It is however important to remember that (148,149) are not the full result, but only the *non analytic* contribution to the ground state energy. As discussed at length in the main text and in Section C, the ground state energy is also shifted by analytic contributions (in s) related to the non-harmonicity of the potential.

For $q = 1$, we have demonstrated that the first analytic contributions vanish (up to $\mathcal{O}(s^2)$) and observed numerically that this is true at all orders in s . Hence the final result for $q = 1$ is

$$\mathcal{E}_1(1) + \sqrt{|E|} \simeq -\alpha_1 \frac{D}{|E|} e^{-4|E|^{3/2}/(3D)} \quad (151)$$

not only the power law of the pre-exponential term perfectly agrees with the numerical result (see letter and section D), but moreover the dimensionless factor $\alpha_1 \simeq$

0.04078 coincides with the value extracted numerically. Nonetheless, we recall that the generalised Lyapunov exponent $\Lambda(q)$ is not related to this eigenvalue, but is controlled by the largest eigenvalue $-\mathcal{E}_0(q)$ of the operator \mathcal{O}_q .

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 [19] although it is not the standard one for the depinning problem, where the elastic line is usually free to move, it is a usual setting in the related sandpile problem